Analytical Treatment of Some Nonlinear Beam Dynamics Problems in Storage Rings

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In this paper we treat some nonlinear beam dynamics problems in storage rings, such as beam dynamic apertures of multipoles and wiggles, beam-beam effects, nonlinear space charge effect, and nonlinear electron cloud effect, analytically. The corresponding analytical expressions developed in this paper are useful both in understanding the physics behind these problems and also in making practical quick hand estimations.

1. Introduction

In storage rings many physical phenomena connected with particles' motions are caused by the nonlinear forces, either static or dynamic, acting on the moving particles. Among them, one finds dynamic apertures limited by static multipoles and wigglers, beam-beam effects due to dynamic nonlinear beam-beam interaction forces, nonlinear space charge and electron cloud effects, which are separately treated in the following sections.

2. Dynamic apertures of multipoles

We start with the simplest case, which is the physical and methmatical bases for the analytical treating of other differnt subjects in the other sections, i.e., the dynamic aperture limited by a single nonlinear multipole located somewhere inside a storage ring. The Hamiltonian of this problem is expressed as follows

$$H = \frac{p^2}{2} + \frac{K(s)}{2}x^2 + \frac{1}{m!B_0\rho}\frac{\partial^{m-1}B_z}{\partial x^{m-1}}x^m L\sum_{k=-\infty}^{\infty}\delta(s-kL) \tag{1}$$

with

$$B_z = B_0 x^{m-1} b_{m-1} \tag{2}$$

where ρ is the bending radius corresponding to B_0, and L is the circumference of the ring. The general formula for the dynamic aperture limited by this multipole reads [1]

$$A_{dyna,2m,x} = \sqrt{2\beta_x(s)}\left(\frac{1}{m\beta_x^m(s(2m))}\right)^{\frac{1}{2(m-2)}} \times \left(\frac{\rho}{|b_{m-1}|L}\right)^{1/(m-2)} \tag{3}$$

where $s(2m)$ is the location of this multipole. The dynamic aperture in vertical plane could be estimated as

$$A_{dyna,2m,y} = \sqrt{\frac{\beta_x(s(2m))}{\beta_y(s(2m))}(A_{dyna,2m,x}^2 - x^2)} \tag{4}$$

where $\beta_y(s(2m))$ is the vertical beta function where the multipole is located. If the are many independent multipoles, one can estimate their combined effects through following equation

$$A_{dyna,total} = \frac{1}{\sqrt{\sum_{i,m}\frac{1}{A_{dyna,2m,i}^2}}} \tag{5}$$

The validity of eqs. 3, 4, and 5 has been checked with numerical simulation results [1].

3. Dynamic aperture limited by wigglers

Considering a wiggler of sinusoidal magnetic field variation, one can express the wiggler's magnetic fields, which satisfies Maxwell equations, as follows

$$B_x = \frac{k_x}{k_y}B_0\sinh(k_x x)\sinh(k_y y)cos(ks) \tag{6}$$

$$B_y = B_0\cosh(k_x x)\cosh(k_y y)cos(ks) \tag{7}$$

$$B_z = -\frac{k}{k_y}B_0\cosh(k_x x)\sinh(k_y y)\sin(ks) \tag{8}$$

with

$$k_x^2 + k_y^2 = k^2 = \left(\frac{2\pi}{\lambda_w}\right)^2 \tag{9}$$

where B_0 is the peak sinusoidal wiggler magnetic field, λ_w is the period length of the wiggler, and x, y, s represent horizontal, vertical, and beam moving directions, respectively.

The Hamiltonian describing particle's motion can be written as

$$H_w = \frac{1}{2}\left(p_z^2 + (p_x - A_x\sin(ks))^2 + (p_y - A_y\sin(ks))^2\right) \tag{10}$$

where

$$A_x = \frac{1}{\rho_w k}\cosh(k_x x)\cosh(k_y y) \tag{11}$$

$$A_x = -\frac{k_x}{k_y}\frac{\sinh(k_x x)\sinh(k_y y)}{\rho_w k} \quad (12)$$

and ρ_w is the radius of curvature of the wiggler peak magnetic field B_0, and $\rho_w = E_0/ecB_0$ with E_0 being the electron energy. After making a canonical transformation to betatron variables, averaging the Hamiltonian over one period of wiggler, and expanding the hyperbolic functions to the fourth order in x and y, one gets

$$\mathcal{H}_w = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{4k^2\rho_w^2}(k_x^2x^2 + k_y^2y^2)$$

$$+\frac{1}{12k^2\rho_w^2}(k_x^4x^4 + k_y^4y^4 + 3k^2k_x^2x^2y^2)$$

$$-\frac{\sin(ks)}{2k\rho_w}\left(p_x(k_x^2x^2 + k_y^2y^2) - 2k_x^2p_yxy\right) \quad (13)$$

Now we insert a "wiggler" of only one period (or one cell) into a storage ring located at s_w. The total Hamiltonian of the ring in the vertical plane can be expressed as follows

$$H = H_0 + \frac{1}{4\rho^2}y^2 + \frac{k_y^2}{12\rho^2}y^4\lambda_w\sum_{i=-\infty}^{\infty}\delta(s - iL) \quad (14)$$

where H_0 is the Hamiltonian without the inserted wiggler, L is the circumference of the ring, and $k_y = k$. It is obvious that the perturbation is a delta function octupole. Comparing eq. 1 with eq. 14, by analogy, one finds easily that

$$\frac{b_3}{\rho}L = \frac{k_y^2\lambda_w}{3\rho_w^2} \quad (15)$$

and the dynamic aperture limited by this one period "wiggler" as

$$A_{1,y}(s) = \frac{\sqrt{\beta_y(s)}}{\beta_y(s_w)}\left(\frac{3\rho_w^2}{k_y^2\lambda_w}\right)^{1/2} \quad (16)$$

where $\beta_y(s)$ is the unperturbed beta function. In fact, a wiggler is an insertion device which is composed of a large number of cells, say, N_w, and the wiggler length $L_w = N_w\lambda_w$. Now, the first question which follows is what the combined effect of these N_w cells will be. According to ref. [1], one has

$$\frac{1}{A_{N_w,y}^2(s)} = \sum_{i=1}^{N_w}\frac{1}{A_{i,y}^2} = \sum_{i=1}^{N_w}\left(\frac{k_y^2}{3\rho_w^2\beta_y(s)}\right)\beta_y^2(s_{i,w})\frac{L_w}{N_w} \quad (17)$$

where the index i indicates different cell. When N_w is a large number, Eq. 17 can be simplified as:

$$\frac{1}{A_{N_w,y}^2(s)} = \frac{k_y^2}{3\rho_w^2\beta_y(s)}\int_{s_{w_0}-L_w/2}^{s_{w_0}+L_w/2}\beta_y^2(s)ds \quad (18)$$

where s_{w_0} correspond to the center of the wiggler. To be practical, one could replace $\beta_y^2(s)$ inside the integral by $\beta_{y,m}^2$ which is the beta function value in the middle of the wiggler, and one gets

$$A_{N_w,y}(s) = \sqrt{\frac{3\beta(s)}{\beta_{y,m}^2}}\frac{\rho_w}{k_y\sqrt{L_w}} \quad (19)$$

$$A_{N_w,x}(s) = \sqrt{\frac{\beta_y(s)}{\beta_x(s)}(A_{N_w,y}(s)^2 - y^2)} \quad (20)$$

If there are more than one wigglers in a storage ring, the total dynamic aperture limited by these wigglers can be estimated by applying eq. 5.

Eq. 19 has been checked with numerical simulation results [2].

4. Beam-beam effects and limitations

For two head-on colliding bunches, the incoherent kick felt by each particle can be calculated as

$$\delta y' + i\delta x' = -\frac{N_e r_e}{\gamma_*}f(x, y, \sigma_x, \sigma_y) \quad (21)$$

where x' and y' are the horizontal and vertical slopes, N_e is the particle population in the bunch, r_e is the electron classical radius (2.818×10^{-15} m), σ_x and σ_y are the standard deviations of the transverse charge density distribution of the counter-rotating bunch at IP, γ_* is the normalized particle's energy, and $*$ denotes the test particle and the bunch to which the test particle belongs. When the bunch is Gaussian $f(x, y, \sigma_x, \sigma_y)$ can be expressed by Basseti-Erskine formula

$$f(x, y, \sigma_x, \sigma_y) = \sqrt{\frac{2\pi}{\sigma_x^2 - \sigma_y^2}} \times w\left(\frac{x + iy}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}}\right)$$

$$-\sqrt{\frac{2\pi}{\sigma_x^2 - \sigma_y^2}}\times\exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}\right)w\left(\frac{\frac{\sigma_y}{\sigma_x}x + i\frac{\sigma_x}{\sigma_y}y}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}}\right) \quad (22)$$

where w is the complex error function expressed as

$$w(z) = \exp(-z^2)(1 - \mathrm{erf}(-iz)) \quad (23)$$

For the round beam (RB) and the flat beam (FB) cases one has the incoherent beam-beam kicks expressed as [3]

$$\delta r'[RB] = -\frac{2N_e r_e}{\gamma_* r}\left(1 - \exp\left(-\frac{r^2}{2\sigma^2}\right)\right) \quad (24)$$

$$\delta x'[FB] = -\frac{2\sqrt{2}N_e r_e}{\gamma_* \sigma_x}\exp\left(-\frac{x^2}{2\sigma_x^2}\right)\int_0^{\frac{x}{\sqrt{2}\sigma_x}}\exp(u^2)du \tag{25}$$

$$\delta y'[FB] = -\frac{\sqrt{2\pi}N_e r_e}{\gamma_* \sigma_x}\exp\left(-\frac{x^2}{2\sigma_x^2}\right)\mathrm{erf}\left(\frac{y}{\sqrt{2}\sigma_y}\right) \tag{26}$$

where $r = \sqrt{x^2 + y^2}$. Now we want to calculate the average kick felt by the test particle since the probability to find the transverse displacement of the test particle is not constant (in fact, the probability function is the same as the charge distribution of the bunch to which the test particle belongs in lepton machines due to synchrotron radiations). In the following we assume that the transverse sizes for the two colliding bunches at IP are exactly the same. For the round beam case after averaging one gets

$$\delta \bar{r}'[RB] = -\frac{2N_e r_e}{\gamma_* \bar{r}}\left(1 - \exp\left(-\frac{\bar{r}^2}{4\sigma^2}\right)\right) \tag{27}$$

Although this expression is the same as that of the coherent beam-beam kick for round beams, one should keep in mind that we are not finding coherent beam-beam kick originally, and the difference will be obvious when we treat the vertical motion in the case of flat beams. For the flat beam case, we will treat the horizontal and vertical planes separately. As far as the horizontal kick is concerned, the horizontal kick depends only on one displacement variable just similar to the round beam case, we will use its coherent form expressed as follows

$$\delta x'[FB] = -\frac{2N_e r_e}{\gamma_* \sigma_x}\exp\left(-\frac{x^2}{4\sigma_x^2}\right)\int_0^{\frac{x}{2\sigma_x}}\exp(u^2)du \tag{28}$$

As for the vertical kick, however, one has to make an average over eq. 26 with the horizontal probability distribution function of the test particle, and one has

$$\delta y'[FB] = -\frac{\sqrt{2\pi}N_e r_e}{\gamma_* \sigma_x} < \exp\left(-\frac{x^2}{2\sigma_x^2}\right) >_x \mathrm{erf}\left(\frac{y}{\sqrt{2}\sigma_y}\right) \tag{29}$$

where $<>_x$ means the average over the horizontal probability distribution function of the test particle, and for two identical colliding Gaussian beams $<>_x = 1/\sqrt{2}$. It is obvious that eq. 29 is not the expression for the coherent beam-beam kick. The average over eqs. 24 and 26 is only a technical operation to simplify (or to make equivalence) a two dimensional problem to a one dimensional one. To study both round and flat beam cases, we expand $\delta\bar{r}'$ at $x = 0$ (for round beam we study only vertical plane since the formalism in the horizontal plane is the same), $\delta x'$ and $\delta y'$ expressed in eqs. 27, 28 and 29, respectively, into Taylor series

$$\delta y'[RB] = \frac{N_e r_e}{\gamma_*}\left(\frac{1}{2\sigma^2}y - \frac{1}{16\sigma^4}y^3 + \frac{1}{192\sigma^6}y^5 - \frac{1}{3072\sigma^8}y^7 + \frac{1}{61440\sigma^{10}}y^9 - \frac{1}{1474560\sigma^{12}}y^{11} + \cdots\right) \tag{30}$$

$$\delta_x'[FB] = -\frac{N_e r_e}{2\gamma_*}\left(\frac{2}{\sigma_x^2}x - \frac{1}{3\sigma_x^4}x^3 + \frac{1}{30\sigma_x^6}x^5 - \frac{1}{420\sigma_x^8}x^7 + \frac{1}{7560\sigma_x^{10}}y^9 - \frac{1}{166320\sigma_x^{12}}x^{11} + \cdots\right) \tag{31}$$

$$\delta_y'[FB] = -\frac{N_e r_e}{\sqrt{2}\gamma_*}\left(\frac{2}{\sigma_x\sigma_y}y - \frac{1}{3\sigma_x\sigma_y^3}y^3 + \frac{1}{20\sigma_x\sigma_y^5}y^5 - \frac{1}{168\sigma_x\sigma_y^7}y^7 + \frac{1}{1728\sigma_x\sigma_y^9}y^9 - \frac{1}{21120\sigma_x\sigma_y^{11}}y^{11} + \cdots\right) \tag{32}$$

The differential equations of the motion of the test particle in the transverse planes can be expressed as

$$\frac{d^2y}{ds^2} + K_y(s)y = -\frac{N_e r_e}{\gamma_*}\left(\frac{1}{2\sigma^2}y - \frac{1}{16\sigma^4}y^3 + \frac{1}{192\sigma^6}y^5 - \frac{1}{3072\sigma^8}y^7 + \frac{1}{61440\sigma^{10}}y^9 - \frac{1}{1474560\sigma^{12}}y^{11} + \frac{1}{41287680\sigma^{14}}y^{13} - \cdots\right)\sum_{k=-\infty}^{\infty}\delta(s - kL) \quad \text{(RB)} \tag{33}$$

$$\frac{d^2x}{ds^2} + K_x(s)x = -\frac{N_e r_e}{2\gamma_*}\left(\frac{2}{\sigma_x^2}x - \frac{1}{3\sigma_x^4}x^3 + \frac{1}{30\sigma_x^6}x^5 - \frac{1}{420\sigma_x 8}x^7 + \frac{1}{7560\sigma_x^{10}}x^9 - \frac{1}{166320\sigma_x^{12}}x^{11} + \frac{1}{4324320\sigma_x^{14}}x^{13} - \cdots\right)\sum_{k=-\infty}^{\infty}\delta(s - kL) \quad \text{(FB)} \tag{34}$$

$$\frac{d^2y}{ds^2} + K_y(s)y = -\frac{N_e r_e}{\sqrt{2}\gamma_*}\left(\frac{2}{\sigma_x\sigma_y}y - \frac{1}{3\sigma_x\sigma_y^3}y^3 + \frac{1}{20\sigma_x\sigma_y^5}y^5 - \frac{1}{168\sigma_x\sigma_y^7}y^7 + \frac{1}{1728\sigma_x\sigma_y^9}y^9 - \frac{1}{21120\sigma_x\sigma_y^{11}}y^{11} + \frac{1}{299520\sigma_x\sigma_y^{13}}y^{13} - \cdots\right)\sum_{k=-\infty}^{\infty}\delta(s - kL) \quad \text{(FB)} \tag{35}$$

where $K_x(s)$ and $K_y(s)$ describe the linear focusing of the lattice in the horizontal and vertical planes. The corresponding Hamiltonians are expressed as

$$H = \frac{p_y^2}{2} + \frac{K_y(s)}{2}y^2 + \frac{N_e r_e}{\gamma_*}(\frac{1}{4\sigma^2}y^2 - \frac{1}{64\sigma^4}y^4 + \frac{1}{1152\sigma^6}y^6$$

$$-\frac{1}{24576\sigma^8}y^8 + \cdots)\sum_{k=-\infty}^{\infty}\delta(s-kL) \qquad \text{(RB)} \quad (36)$$

$$H_x = \frac{p_x^2}{2} + \frac{K_x(s)}{2}x^2 + \frac{N_e r_e}{2\gamma_*}(\frac{1}{\sigma_x^2}x^2 - \frac{1}{12\sigma_x^4}x^4 + \frac{1}{180\sigma_x^6}x^6$$

$$-\frac{1}{3360\sigma_x^8}x^8 + \cdots)\sum_{k=-\infty}^{\infty}\delta(s-kL) \qquad \text{(FB)} \quad (37)$$

$$H_y = \frac{p_y^2}{2} + \frac{K_y(s)}{2}y^2$$

$$+\frac{N_e r_e}{\sqrt{2}\gamma_*}(\frac{1}{\sigma_x\sigma_y}y^2 - \frac{1}{12\sigma_x\sigma_y^3}y^4 + \frac{1}{120\sigma_x\sigma_y^5}y^6$$

$$-\frac{1}{1344\sigma_x\sigma_y^7}y^8 + \cdots)\sum_{k=-\infty}^{\infty}\delta(s-kL) \qquad \text{(FB)} \quad (38)$$

where $p_x = dx/ds$ and $p_y = dy/ds$.

Using the general knowledge obtained in section II and comparing the eq. 1 with the Hamiltonians for beam-beam interactions, we have derived beam-beam effect limited beam lifetimes for a rigid flat beam [3]

$$\tau_{bb,y,flat} = \frac{\tau_y}{2}\left(\frac{3}{\sqrt{2}\pi\xi_y}\right)^{-1}\exp\left(\frac{3}{\sqrt{2}\pi\xi_y}\right) \qquad (39)$$

$$\tau_{bb,x,flat} = \frac{\tau_x}{2}\left(\frac{3}{\pi\xi_x}\right)^{-1}\exp\left(\frac{3}{\pi\xi_x}\right) \qquad (40)$$

and a rigid round beam

$$\tau_{bb,y,round} = \frac{\tau_y}{2}\left(\frac{4}{\pi\xi_x}\right)^{-1}\exp\left(\frac{4}{\pi\xi_x}\right) \qquad (41)$$

From eqs. 39 and 40 one finds that for the same $\tau_{y,bb,flat}/\tau_y$, $\tau_{x,bb,flat}/\tau_x$, and $\tau_{y,bb,round}/\tau_y$, one has $\xi_{x,flat} = \sqrt{2}\xi_{y,flat}$, and $\xi_{y,round} = \frac{4\sqrt{2}}{3}\xi_{y,flat} = 1.89\xi_{y,flat}$.

In reality, the colliding bunch is not rigid, the transverse emittance will increase due to the additional heating. In the following we will show how emittance blow-up is included into the beam-beam lifetime expressions.

In e^+e^- storage ring colliders, due to strong quantum excitation and synchrotron damping effects, the particles are confined inside a bunch. The state of the particles can be regarded as a gas, where the positions of the particles follow statistic laws. When two bunches undergo collision at an interaction point (IP, denoted by "*") the particles in each bunch will suffer from additional heatings. Taking the vertical plane for example, one has beam-beam induced kicks in y and $y' = dy/ds$ expressed as

$$\delta y = -\frac{\sigma_s}{f_y}y \qquad (42)$$

$$\delta y' = -\frac{y}{f_y}y \qquad (43)$$

$$\frac{1}{f_y} = \frac{2N_e r_e}{\gamma\sigma_{y,*,+}(\sigma_{x,*,+} + \sigma_{y,*,+})} \qquad (44)$$

where σ_s is the bunch length, N_e is the particle number inside the bunch, r_e is the electron classical radius, $\sigma_{x,*,+}$ and $\sigma_{y,*,+}$ are bunch transverse dimensions just before the two colliding bunches overlapping each other, and $\sigma_{x,*}$ and $\sigma_{y,*}$ are defined as the transverse dimensions when the two bunches are fully overlapped at IP. The invariant of vertical betatron motion can be expressed as [6]

$$a_y^2 = \frac{1}{\beta_y^*}\left(y_*^2 + \left(\beta_{y,*}y_*' - \frac{1}{2}\beta_{y,*}'y_*\right)^2\right) \qquad (45)$$

From eqs. 42 and 43 one finds that

$$\delta a_y^2 = \frac{1}{\beta_{y,*}}\left(\frac{\sigma_s}{f_y}\right)^2 y_*^2\left(1 + \left(\frac{\beta_{y,*}}{\sigma_s}\right)^2\right) \qquad (46)$$

where y_* is the vertical displacement of the test particle with respect to the center of the colliding bunch. Due to the gaseous nature of the particles, one has to take an average of all possible values of y_* according to its statistical distribution function, and from eq. 46 one obtains

$$<\delta a^2> = \frac{1}{\beta_{y,*}}\left(\frac{\sigma_s\sigma_{y,*}}{f_y}\right)^2\left(1 + \left(\frac{\beta_{y,*}}{\sigma_s}\right)^2\right) \qquad (47)$$

The resultant particles' vertical dimension combining the synchrotron radiation and beam-beam effects can be expressed as follows

$$\sigma_{y,*}^2 = \frac{1}{4}\tau_y\beta_{y,*}Q_y$$

$$+\frac{1}{4}\tau_y\beta_{y,*}\left(\frac{1}{T_0\beta_{y,*}}\left(\frac{\sigma_s\sigma_{y,*}}{f_y}\right)^2\left(1 + \left(\frac{\beta_{y,*}}{\sigma_s}\right)^2\right)\right) \qquad (48)$$

where T_0 is the revolution time, τ_y is the radiation damping time, and Q_y is defined according to ref. [6] as $\sigma^2_{y,*,0} = \frac{1}{4}\tau_y\beta_{y,*}Q_y$ with $\sigma_{y,*,0}$ being bunch natural vertical dimension at IP. Solving eq. 48, one finds

$$\sigma^2_{y,*} = \frac{\sigma^2_{y,*,0}}{\left(1 - \frac{\tau_y}{4T_0}\left(\frac{e^2 N_e K_{bb,y}}{E_0}\right)^2\right)} \tag{49}$$

where E_0 is particles' energy, and

$$K_{bb,y} = \frac{\sigma_s}{2\pi\epsilon_0\sigma_{y,*,+}(\sigma_{x,*,+} + \sigma_{y,*,+})}$$

$$\times\left(1 + \left(\frac{\beta_{y,*,+}}{\sigma_s}\right)^2\right)^{1/2} \tag{50}$$

Since $\sigma_y(s) = \sqrt{\epsilon_y\beta_y(s)}$, from eq. 49 one gets

$$\epsilon_y = \frac{\epsilon_{y,0}}{\left(1 - \frac{\tau_y}{4T_0}\left(\frac{e^2 N_e K_{bb,y}}{E_0}\right)^2\right)} \tag{51}$$

where $\epsilon_{y,0}$ is the natrual transverse emittance. For a flat bunch ($\sigma_{y,*,+} << \sigma_{x,*,+}$), from eq. 51 one knows that

$$\sigma_{x,*,+}\sigma_{y,*,+} > \left(\frac{3RN_{IP}(e^2 f N_e\beta_{y,*})^2}{8\pi^2\epsilon_0 m_0 c^2\gamma^5}\right)^{1/2} \tag{52}$$

Defining

$$H = \frac{\sigma_{x,*,+}\sigma_{y,*,+}}{\sigma_{x,*}\sigma_{y,*}} \tag{53}$$

where H is a measure of the plasma pinch effect, assuming that H can be expressed as follows

$$H = \frac{H_0}{\sqrt{\gamma}} \tag{54}$$

and recalling the beam-beam parameter definition

$$\xi_y = \frac{N_e r_e\beta_{y,*}}{2\pi\gamma\sigma_{y,*}(\sigma_{x,*} + \sigma_{y,*})} \tag{55}$$

where β^*_y is the beta function value at the interaction point, σ^*_x and σ^*_y are the bunch transverse dimensions after the plasma pinch effect, respectively, and finally, by combining eqs. 52, 54 and 55 one gets in general case

$$\xi_y \leq \xi_{y,max,em,flat} = \frac{H_0}{2\pi F}\sqrt{\frac{T_0}{\tau_y\gamma N_{IP}}} \tag{56}$$

or for isomagnetic case

$$\xi_y \leq \xi_{y,max,em,flat} = \frac{H_0\gamma}{F}\sqrt{\frac{r_e}{6\pi RN_{IP}}} \tag{57}$$

where $H_0 \approx 2845$, R is the local dipole bending radius, and F is expressed as follows

$$F = \frac{\sigma_s}{\sqrt{2}\beta_{y,*}}\left(1 + \left(\frac{\beta_{y,*}}{\sigma_s}\right)^2\right)^{1/2} \tag{58}$$

The subscript em in eqs. 56 and 57 denotes the emittance blow-up limited beam-beam parameter. When $\sigma_s = \beta_{y,*}$ one has $F = 1$.

Now taking into account of the emittance blow-up effect due to beam-beam interactions, in a heuristic way, one gets

$$\tau_{bb,y,flat} = \frac{\tau_y}{2}\left(\frac{3\xi_{y,max,em,flat}}{\sqrt{2}\pi\xi_{y,max,0}\xi_y N_{IP}}\right)^{-1}$$

$$\times\exp\left(\frac{3\xi_{y,max,em,flat}}{\sqrt{2}\pi\xi_{y,max,0}\xi_y N_{IP}}\right) \tag{59}$$

and

$$\tau_{bb,y,round} = \frac{\tau_y}{2}\left(\frac{3\xi_{y,max,em,round}}{\sqrt{2}\pi\xi_{y,max,0}\xi_y N_{IP}}\right)^{-1}$$

$$\times\exp\left(\frac{3\xi_{y,max,em,round}}{\sqrt{2}\pi\xi_{y,max,0}\xi_y N_{IP}}\right) \tag{60}$$

with

$$\xi_{y,max,em,round} = 1.89\xi_{y,max,em,flat} \tag{61}$$

where $\xi_{y,max,0}$ is rigid beam case limiting value. Taking $\xi_{y,max,0} = 0.0447$ means that we quantify the term "beam-beam limit" for the beam-beam limited beam lifetime being one hour at $\tau_y = 30$ ms with $N_{IP} = 1$.

Eqs. 56 and 59 have been checked with some machine operation results [4].

5. Beam-beam effects with crossing angle

To get a higher luminosity one could run a circular collider in the multibunch operation mode with a definite collision crossing angle. Different from the head-on collision discussed above, the transverse kick received by a test particle due to the space charge field of the counter rotating bunch will depend on its longitudinal position with respect to the center of the bunch which the test particle belongs to. In this section we consider first a flat beam colliding with another flat beam with a half crossing angle of ϕ in the horizontal plane. Due to the crossing angle the two curvilinear coordinates of the two colliding beams at the interaction point will be no longer coincide. When the crossing angle is not too large one has

$$x^* = x + z\phi \tag{62}$$

where x^* is the horizontal displacement of the test particle to the center of the colliding bunch, z and x are the longitudinal and horizontal displacements of the test particle from the center of the bunch to which it belongs. Now we recall eq. 37 which describes the Hamiltonian of the horizontal motion of a test particle in the head-on collision mode, and by inserting eq. 62 into eq. 37 we get

$$H_x = \frac{p_x^2}{2} + \frac{K_x(s)}{2}x^2 + \frac{N_e r_e}{2\gamma_*}(\frac{1}{\sigma_x^2}(x+z\phi)^2$$

$$-\frac{1}{12\sigma_x^4}(x+z\phi)^4 + \frac{1}{180\sigma_x^6}(x+z\phi)^6$$

$$-\frac{1}{3360\sigma_x^8}(x+z\phi)^8+\cdots)\sum_{k=-\infty}^{\infty}\delta(s-kL) \qquad \text{(FB)} \tag{63}$$

Since the test particle can occupy a definite z within the bunch according to a certain probability distribution, say Gaussian, it is reasonable to replace z in eq. 63 by σ_z, and in this way we reduce a two dimensional Hamiltonian expressed in eq. 63 into a one dimensional one. What should be noted is that eq. 63 takes only the test particle's longitudinal position into consideration which is regarded as a small perturbation to the head-on collision case, and the geometrical effect will included later. To simplify our analysis we consider only the lowest synchro-betatron nonlinear resonance, i.e., $3Q_x \pm Q_s = p$ (where Q_s is the synchrotron oscillation tune, and p is an integer) which turns out to be the most dangerous one. Following the same procedure in section 4 one gets the dynamic aperture due to the lowest synchro-betatron nonlinear resonance as follows

$$A_{syn-beta,x}(s) = \left(\frac{2\beta_x(s)}{3\beta_x(s_{IP})^3}\right)^{1/2}\frac{2\gamma_*\sigma_x^4}{N_e r_e \sigma_z \phi} \tag{64}$$

and

$$\mathcal{R}_{syn-beta,x} = \frac{A_{syn-beta,x}(s)^2}{\sigma_x(s)^2} = \frac{2}{3\pi^2}\left(\frac{1}{\xi_x^*\Phi}\right)^2 \tag{65}$$

where $\Phi = \frac{\sigma_z}{\sigma_x}\phi$ is Piwinski angle. Now we are facing a problem of how to combine the two effects: the principal vertical beam-beam effect and the horizontal crossing angle induced perturbation. To solve this problem we assume that the total beam lifetime due to the vertical and the horizontal crossing angle beam-beam effects can be expressed as

$$\tau^*_{bb,total} = \frac{\tau_x^* + \tau_y^*}{4}\left(\frac{1}{\frac{1}{\mathcal{R}_{y,8,FB}} + \frac{1}{\mathcal{R}_{syn-beta,x}}}\right)^{-1} \times$$

$$\exp\left(\frac{1}{\frac{1}{\mathcal{R}_{y,8,FB}} + \frac{1}{\mathcal{R}_{syn-beta,x}}}\right) \qquad \text{(FB)} \tag{66}$$

where $\mathcal{R}_{y,8,FB} = \frac{3}{\sqrt{2}\pi\xi_y}$ for rigid beam case. To include emittance blow-up effects one should follow the same procedure shown at the end of section IV.

Eq. 66 has been applied to KEK-B low energy ring to estimate the the luminosity reduction due to crossing angle effect [5].

6. Parasitic crossing effects

Parasitic crossings in e^+e^- storage ring colliders such as PEP-II working in by-2 mode will introduce additional beam lifetime limitation together with beam-beam effects at IP with or without crossing angle. If the transverse separation of the two parasitic crossing bunches is $\Sigma_{PC} = \sqrt{d_x^2 + d_y^2}$, with d_x and d_y are separations in horizontal and vertical plane, respectively. According to ref. [7] the beam lifetime limited by one parasitic crossing

$$\tau_{PC,y,RB} = \frac{\tau_y}{2}\left(\mathcal{R}_{y,PC,RB}\right)^{-1}\exp\left(\mathcal{R}_{y,PC,RB}\right)$$

$$= \frac{\tau_y}{2}\left(\frac{4}{\pi\xi_{PC,y}}\right)^{-1}\exp\left(\frac{4}{\pi\xi_{PC,y}}\right) \tag{67}$$

with

$$\xi_{PC,y} = \frac{r_e N_e \beta_{PC,x}}{2\pi\gamma_*\Sigma_{PC}^2} = \frac{r_e N_e \beta_{PC,y}}{2\pi\gamma_* d_x^2} \tag{68}$$

where $\beta_{PC,y}$ is the vertical beta function value at the parasitic crossing point, and d_y has been set to zero as a special case of a horizontal separation. What we should do now is to combine the effects from the beam-beam interactions at IP and PC to obtained the corresponding resultant beam lifetime as follows

$$\tau_{bb,total} = \frac{\tau_y}{2}\left(\mathcal{R}_{total}\right)^{-1}\exp\left(\mathcal{R}_{total}\right) \tag{69}$$

where

$$\mathcal{R}_{total} = \frac{1}{\frac{1}{\mathcal{R}_{y,IP,FB}} + \frac{1}{\mathcal{R}_{y,PC,RB}}} \tag{70}$$

$$\mathcal{R}_{y,IP,FB} = \frac{3}{\sqrt{2}\pi\xi_y} \tag{71}$$

$$\mathcal{R}_{y,PC,RB} = \frac{4}{\pi\xi_{PC,y}} \tag{72}$$

If there are N_{PC} parasitic crossings per turn, eq. 71 should be replaced by

$$\mathcal{R}_{total} = \frac{1}{\frac{1}{\mathcal{R}_{y,IP,FB}} + \sum_{i=1}^{N_{PC}}\frac{1}{\mathcal{R}_{y,PC,RB,i}}} \tag{73}$$

where

$$\mathcal{R}_{y,PC,RB,i} = \frac{4}{\pi \xi_{PC,y,i}} \tag{74}$$

$$\xi_{PC,y,i} = \frac{r_e N_e \beta_{PC,y,i}}{4\pi\gamma_* \Sigma^2_{PC,y,i}} = \frac{r_e N_e \beta_{PC,y,i}}{2\pi\gamma_* d^2_{x,i}} \tag{75}$$

where d_y is set to zero. To include emittance blow-up effects one should follow the same procedure shown at the end of section IV.

Eq. 69 has been applied to the PEP-II low energy ring working in by-2 mode [7].

7. Combined beam-beam and electron cloud effects

Electron clouds produced and trapped by the positron beam in the vacuum chamber can perturb the motion of positrons in return. In this section we focus ourselves to the special case where significant amount of electrons are traped near the positron beam axis with almost the same dimensions as those of trapping positron beam, and the electron-clouds far from the positron beam are not the subject of interests of this section. We define the local electron-cloud and positron beam interaction force as $f'_{ec}(s_0)$, this *differential force* (where $'$ denotes d/ds), can be made equivalent to a virtual local beam-beam force $\mathcal{F}_{bb}(s_0)$. The relation between $f'_{ec}(s_0)$ and $\mathcal{F}_{bb}(s_0)$ can be expressed as

$$f'_{ec}(s_0) = \frac{1}{2L}\mathcal{F}_{bb}(s_0) \tag{76}$$

and the $f'_{ec}(s_0)$ induced differential positron linear tune shift is expressed as

$$\xi'_{ec}(s_0) = \frac{r_e N_{ec} \beta_{+,y}(s_0)}{2\pi\gamma_+ \sigma_{+,y}(s_0)(\sigma_{+,x}(s_0)+\sigma_{+,y}(s_0))}\left(\frac{1}{2L}\right) \tag{77}$$

where $\sigma_{+,x}$ and $\sigma_{+,y}$ are the transverse rms dimensions of the electron-clouds and positron beam, L is the circumference of the storage ring, $\beta_{+,y}$ is the vertical beta function for positrons, γ_+ is the normalized positrons' energy, and finally N_{ec} is total electron-cloud charge numbers around the ring within a transverse cross section of $2\pi\sigma_{+,x}\sigma_{+,y}$. Now one could make use of the analytical results for the beam-beam interactions in an e^+e^- storage ring collider developed in ref. [3] to estimate the vertical dynamic aperture limited by the differential electron-cloud nonlinear forces

$$\left(\frac{\sigma_{+,y}(s_0)}{A'_{ec,y}(s_0)}\right)^2 = \frac{N_{ec} r_e \beta_y(s_0)}{6\sqrt{2}\gamma_+ \sigma_{+,x}(s_0)\sigma_{+,y}(s_0)L} \tag{78}$$

The total contribution of the electron-cloud around the ring to the vertical dynamic aperture can be estimated according to ref. [1] as

$$\left(\frac{\sigma_{+,y}}{A_{ec,y}}\right)^2 = \int_{s_0}^{s_0+L} \frac{N_{ec} r_e \beta_y(s_0)}{6\sqrt{2}\gamma_+ \sigma_{+,x}(s_0)\sigma_{+,y}(s_0)L} ds_0 \tag{79}$$

One finds that

$$\mathcal{R}^2_{ec,y} = \left(\frac{A_{ec,y}}{\sigma_{+,y}}\right)^2 \approx \frac{3\sqrt{2}\gamma_+}{\pi r_e \beta_{av,y}\rho_{ec} L} \tag{80}$$

where $\beta_{av,y}$ is the average vertical beta function around the ring, and ρ_{ec} is the average electron-cloud density inside the vacuum chamber which is defined as follows:

$$\rho_{ec} = \frac{N_{ec}}{2\pi\sigma_{av,+,x}\sigma_{av,+,y}L} \tag{81}$$

where $\sigma_{av,+,x}$ and $\sigma_{av,+,y}$ are the average beam transverse dimensions around the ring. The total normalized vertical dynamic aperture limited together by the beam-beam and the electron-cloud effects can be obtained as

$$\mathcal{R}^2_{total,+,y} = \frac{1}{\frac{1}{\mathcal{R}^2_{bb,+,y}} + \frac{1}{\mathcal{R}^2_{ec,y}}} \tag{82}$$

with $\mathcal{R}^2_{bb,+,y}$ expressed as

$$\mathcal{R}^2_{bb,+,y} = \left(\frac{A_{bb,y,IP}}{\sigma_{+,y,IP}}\right)^2 = \frac{3}{\sqrt{2}\pi\xi_{bb,+,y}} \tag{83}$$

where $\xi_{bb,+,y}$ is the linear beam-beam tune shift of the positron beam in the vertical plane, and the subscript IP denotes the interaction point. The positron's lifetime due to the combined beam-beam and electron-cloud effects can be estimated as:

$$\tau_{total,+,y} = \frac{\tau_{+,y}}{2}\left(\mathcal{R}^2_{total,+,y}\right)^{-1}\exp\left(\mathcal{R}^2_{total,+,y}\right) \tag{84}$$

where $\tau_{+,y}$ is the damping time of positron in the vertical plane.

8. Nonlinear space charge effect

Considering an electron storage ring, particles inside a bunch will subject to collective space charge force from the bunch. As we will show later, in some special situations, the effect coming from this force could not be neglected. We start with the linear incoherent space charge tune shift of the machine at the center of the bunch

$$\xi_{sc,y} = -\frac{r_e N_e \beta_{av,y}}{2\pi\gamma\sigma_y(\sigma_x+\sigma_y)}\left(\frac{L}{\sqrt{2\pi}\beta^2\gamma^2\sigma_z}\right) \tag{85}$$

where N_e is the particle population inside the bunch, σ_z is the bunch length, and $\beta_{av,y}$ is the average over the ring. In fact, as in the previous section, one can define the *differential* space charge tune shift from which the space charge tune shift of the ring can be obtained

$$\xi'_{sc,y}(s_0) = -\frac{r_e N_e \beta_y(s_0)}{2\pi\gamma\sigma_y(s_0)(\sigma_x(s_0)+\sigma_y(s_0))}\left(\frac{1}{\sqrt{2\pi}\beta^2\gamma^2\sigma_z}\right) \tag{86}$$

where $'$ denotes d/ds, s_0 denotes un arbitrary position in the ring. Recalling the expression of the beam-beam tune shift of a storage ring collider, one has

$$\xi_{bb,y}(s_{IP}) = \frac{r_e N_e \beta_{y,IP}}{2\pi\gamma\sigma_y(s_{IP})(\sigma_x(s_{IP})+\sigma_y(s_{IP}))} \tag{87}$$

where s_{IP} denotes the interaction point. Comparing eq. 86 with eq. 87, one finds that the transverse deflecting forces from the differential space charge and the beam-beam interactions have the following relation

$$f'_{sc}(s) = f_{bb}(s_{IP})G \tag{88}$$

with

$$G = -\left(\frac{1}{\sqrt{2\pi}\beta^2\gamma^2\sigma_z}\right) \tag{89}$$

where f'_{sc} and f_{bb} are the total transverse forces including, of course, nonlinear parts. We conclude that the differential space charge effect can be made equivalent to the problem of beam-beam interaction in an storage ring collider.

By analogy one knows the dynamic aperture determined by the nonlinear (octupole is the lowest nonlinear multipole) differential space charge force

$$\left(A_{sc,y}(s)^2\right)' = \frac{\beta_y(s)}{\beta_y(s_0)^2}\left(\frac{3\sqrt{2}\gamma\sigma_x(s_0)\sigma_y^3(s_0)}{N_e r_e G}\right) \quad \text{(FB)} \tag{90}$$

The total dynamic aperture limited by the space charge force can be calculated as

$$A_{total,sc,y}(s) = \frac{1}{\sqrt{\sum_{s_0=0}^{L}\frac{1}{(A_{sc,y}(s)^2)'}}} \tag{91}$$

$$\frac{1}{A^2_{total,sc,y}(s)} =$$

$$\int_{s_0=0}^{L}\frac{\beta_y(s_0)^2}{\beta_y(s)}\left(\frac{N_e r_e}{6\sqrt{\pi}\beta^2\gamma^3\sigma_x(s_0)\sigma_y(s_0)^3\sigma_z}\right)ds_0 \tag{92}$$

where the differential space charge forces are assumed to be independent. After some mathematical simplification and using eq. 85, one gets

$$\mathcal{R}_y^2 = \left(\frac{A_{total,sc,y}(s)}{\sigma_y(s)}\right)^2 = \frac{3}{\sqrt{2\pi}\xi_{sc}} \tag{93}$$

The particle's lifetime due to nonlinear space charge forces can be estimated as:

$$\tau_{sc,y}(\xi_{sc,y}) = \frac{\tau_y}{2}\left(\mathcal{R}_y^2\right)^{-1}\exp\left(\mathcal{R}_y^2\right)$$

$$= \frac{\tau_y}{2}\left(\frac{3}{\sqrt{2\pi}\xi_{sc,y}}\right)^{-1}\exp\left(\frac{3}{\sqrt{2\pi}\xi_{sc,y}}\right) \tag{94}$$

Knowing the particle's lifetime limited by the nonlinear space charge force expressed in eq. 94, one can calculate the relative particle's survival population, $R(\xi_{sc,y})$, at the moment of ejection ($t = \tau_{st}$) by the following formula

$$R(\xi_{sc}) = \exp\left(-\frac{\tau_{st}}{\tau_{sc,y}(\xi_{sc,y})}\right) \tag{95}$$

Now we apply eq. 95 to TESLA damping ring [8] with $\tau_y = 28$ ms, and storage time $\tau_{st} = 200$ ms, and calculate the relative particle's survival population with respect to the the linear space charge tune shift $\xi_{sc,y}$. From eq. 95 one finds that to avoid the particle loss due to nonlinear space charge forces, one has to choose $\xi_{sc,y}$ below 0.07 (less than 1% particles are lost), which coincides with the conclusion from the numerical simulations in ref. [8] which states clearly that the condition $\xi_{sc,y} < 0.1$ should be fulfilled. Taking the TESLA parameters, $E_0 = 5$ GeV, $L = 17$ km, $N_e = 2 \times 10^{10}$, $\sigma_z = 6$ mm, and the normalized transverse emittances, $\epsilon_{x,n} = 9 \times 10^{-6}$ mrad and $\epsilon_{y,n} = 2 \times 10^{-8}$ mrad, one finds $\xi_{sc,y} = 0.248$ and $R(\xi_{sc,y}) = 7.7\%$, which are intolerable. In order to solve this problem, instead of increasing the damping ring's energy, a method has been proposed in ref. [8] to increase the beam dimensions in the long straight sections of the "Dog-Bone" type damping ring by using screw quadrupoles, which has reduced the space charge tune shift well below the threshold, $\xi_{sc,y} = 0.1$.

9. Conclusion

Many complex phenomena in storage rings are connected with nonlinear beam dynamics, such as the subjects treated in this paper. Together with experiments and numerical simulations, analytical treatment plays an important role in understanding the relevant physical processes and is very helpful in designing and operating machines.

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