$B \rightarrow X_s \gamma$ at NNLO: 

effect of the photon energy cut

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Fermilab

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Outline

• Introduction
  • Experimental result, theory at NLO
• Status of NNLO calculation
  • Matching, running, matrix elements
• Photon energy cut
  • Factorization and resummation
  • Two-loop calculation of the soft- and jet-function
  • A wonderful formula
B → X_s γ

- The mother of all FCNC processes
  - Suppressed in the SM, but large enough to be well measured.
  - Sensitive probe of New Physics. E.g. MSSM
$\text{Br}_{\exp}(E_{\gamma} > 1.6\text{GeV}) = (3.55 \pm 0.24^{+0.09}_{-0.1} \pm 0.03) \times 10^{-4}$

HFA'06

$m_c/m_b = 0.29 \pm 0.02$ (pole mass)

$m_c/m_b = 0.22 \pm 0.04$ (MS)

$\text{Br}_{\text{SM}}(E_{\gamma} > 1.6\text{GeV}) = (3.33 \pm 0.29) \times 10^{-4}$, $m_c/m_b = 0.26 \pm 0.01$

Haisch '06
**Experiment?**

\[ \text{Br}_{\exp}(E_{\gamma} > 1.6\text{GeV}) = (3.55 \pm 0.24^{+0.09}_{-0.1} \pm 0.03) \times 10^{-4} \]

- All experiments have \( E_{\gamma} > E_0 \geq 1.8 \text{ GeV} \)
- HFAG uses *model* shape function to extrapolate results to \( E_0 \geq 1.6 \text{ GeV} \).
  - Use single functional form with two parameters (\( m_b \) & \( \mu_\pi \)) and
  - model falls off exponentially instead of power-like
- Until recently, experiments used to extrapolate to the total rate.
Extrapolation to lower cut energy $E_0$

- Standard OPE calculation is unreliable at high $E_0$. (Note $E_{\text{max}}=m_B/2$)
  - Is it reliable at $E_0=1.6\text{GeV}$?
- For current measurements $E_0 \leq 1.9\text{GeV}$, model independent calculation is possible!
  - Multi-Scale OPE → *third part of the talk*
- Compare event fractions: $F(E_0)=\Gamma(E_\gamma>E_0)/\Gamma_{\text{tot}}$

<table>
<thead>
<tr>
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<th>MSOPE @ NLO Neubert ’04</th>
<th>HFAG ‘06 Buchmüller, Flächer ’06</th>
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<tr>
<td>$F(E_\gamma &gt; 1.8\text{ GeV})$</td>
<td>$0.88\pm0.07$</td>
<td>$0.933\pm0.006$</td>
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PT error dominates

Neubert ‘04
• Experimental uncertainty already somewhat smaller than theoretical uncertainty. Need
  • total rate to NNLO and
  • calculation of event fraction to NNLO.

\[ \text{Br}(E_{\gamma} > E_0) = \text{Br}(\text{tot}) \times F(E_0) \]

• With present agreement between theory and experiment large deviation unlikely.
  • Reliable uncertainties crucial to get meaningful bounds on New Physics.
Total $B \rightarrow X_s \gamma$ rate to NNLO: quite a loopfest in itself!
Calculation of the total rate

Three steps

At lowest order in $1/m_b$ OPE calculation boils down to evaluating the partonic $b \rightarrow s \gamma$ matrix elements

Leading power corrections are known to LO

NLO EW corrections known
Effective weak Hamiltonian

\[ \mathcal{H}_W = \frac{G_F}{\sqrt{2}} \sum_{p=u,c} V_{ps}^* V_{pb} \left[ C_1 Q_1^p + C_2 Q_2^p + \sum_{i=3}^{8} C_i Q_i \right] \]

- Small contribution from \(|C_{3-6}(m_b)| < 0.07\)

**LO**
- \(Q_7\)
  - \(m_b \bar{s}_L \sigma_{\mu\nu} F^{\mu\nu} b_R\)
  - \(C_7(m_b) \approx -0.3\)

**NLO**
- \(Q_8\)
  - \(m_b \bar{s}_L \sigma_{\mu\nu} G^{\mu\nu} b_R\)
  - \(C_8(m_b) \approx -0.15\)
- \(Q_{1,2}\)
  - \(\bar{c}_L \gamma_\mu b_L \bar{s}_L \gamma_\mu c_L\)
  - \(|C_{1,2}(m_b)| \approx 1\)
Effective Hamiltonian at NNLO: matching

Figure 1: One of the $\mathcal{O}(10^3)$ three-loop diagrams that we have calculated.

- Matching at the weak scale complete.
- NNLO = three loops because of weak-interaction loop ("penguin")

Misiak and Steinhauser ‘04
Effective Hamiltonian at NNLO: running

- Done: Four quark operators, self-mixing of dipole operators.

- New: Four-loop anomalous dimensions for mixing $Q_{1-6}$ into $Q_7$. (>100’000 diagrams)
  - Effect on Br is -2.4%, larger than expected.

- Soon: $Q_{1-6}$ into $Q_8$. Completes NNLO eff. Hamiltonian!
Done: Matrix element of $Q_7$.

Most complicated part of entire calculation are the 3-loop matrix elements of $Q_{1,2}$.

- $n_f$-part known.
- Expansion around $m_c/m_b \gg 1$ (!) calculated, extrapolation to $m_c/m_b$. Misiak et al., to appear
- At NLO this works numerically fairly well, but...
1. The total rate cannot be measured.
2. Even if it could be measured, it could not be calculated.
3. Even if it could be calculated, the result would be incorrect.

after Gorgias, 483-375 BC
Photon energy cut

- Total rate is not measurable, need to impose cut on photon energy $E_\gamma > E_0$
  - Experimentally very energetic photon is necessary to suppress background.
  - All experiments have $E_0 \geq 1.8\text{GeV}$.
  - Note: $E_\gamma < m_B/2 \approx 2.6\text{GeV}$
  - Need to cut out charm resonances: decay $B \rightarrow \psi X$ followed by $\psi \rightarrow X\gamma$. Achieved by setting $E_0 > 1.5\text{GeV}$. 
• Belle has $E_0=1.8\text{GeV}$, BaBar $E_0=1.9\text{GeV}$

• Cut complicates theoretical analysis...
Scales

- With a cut $E_Y > E_0$, problem contains three relevant scales
  - Hard scale: $m_b$
  - Jet scale: $M_X \sim (m_b \Delta)^{1/2}$
  - Soft scale: $\Delta = m_b - 2E_0$
- OPE becomes expansion in $\Lambda/\Delta$ instead of $\Lambda/m_b$!

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Factorization theorem

\[ \Gamma \sim H^2 J \otimes S \]

hard jet soft

shape function

Korchemsky, Sterman '94
Event fraction

\[ F(E_0) = h(m_b, \mu) \int_0^\Delta dP \int_0^P d\omega J(m_b(P - \omega), \mu) S(\omega, \mu) \]

- Any choice of scale \( \mu \) leads to large perturbative log’s, either in \( h, J, \) or \( S. \)
  - Evaluate each part at its characteristic scale and use RG evolution to evolve to a common scale.
  - This resums all (Sudakov) logarithms.
- Same jet-function \( J \) as in DIS for \( x \to 1. \)
Calculation of the soft function

\[ s\left(\ln \frac{\Omega}{\mu}, \mu \right) \equiv \int_0^\Omega d\omega \langle b_v| \bar{h}_v \delta(\omega + in \cdot D) h_v |b_v \rangle = \frac{1}{2\pi i} \oint d\omega \langle b_v| \bar{h}_v \frac{1}{\omega + in \cdot D + i0} h_v |b_v \rangle \]

\[ S_{\text{parton}}(\omega, \mu) \]

- Write \( \delta \)-distribution operator in shape function as discontinuity of propagator
  
  Calculation can be done with standard techniques for loop integrals
Calculation of the soft function

- ×-vertex denotes possible insertion of light-cone propagator

- All diagrams are expressed in terms of integrals

\[
\int d^d k d^d l \frac{(-1)^{-a_1-a_2-a_3-b_1-b_2-b_3-c_1-c_2}}{(k^2)^{a_1} (l^2)^{a_2} [(k - l)^2]^{a_3} (v \cdot k)^{b_1} (v \cdot l)^{b_2} [v \cdot (k + l)]^{b_3} (n \cdot k + \omega)^{c_1} (n \cdot l + \omega)^{c_2}}
\]

- Use integration-by-part relations to reduce to four master integrals (“AIR” by Anastasiou and Lazopoulos hep-ph/0404258)
\[ s(L, \mu) = 1 + \frac{\alpha_s(\mu)}{4\pi} \left[ c^{(1)}_0 + 2\gamma_0 L - \Gamma_0 L^2 \right] \]
\[ + \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 \left[ c^{(2)}_0 + \left( 2c^{(1)}_0(\gamma_0 - \beta_0) + 2\gamma_1 + \frac{2\pi^2}{3} \Gamma_0 \gamma_0 + 4\zeta_3 \Gamma_0 \right) L \right] \]
\[ + \left( 2\gamma_0(\gamma_0 - \beta_0) - c^{(1)}_0 \Gamma_0 - \Gamma_1 - \frac{\pi^2}{3} \Gamma_0^2 \right) L^2 + \left( \frac{2}{3} \beta_0 - 2\gamma_0 \right) \Gamma_0 L^3 + \frac{\Gamma_0^2}{2} L^4 \]

- All \( L=\ln(\Omega/\mu) \) terms known from solving RG equation for the shape function → check on the calculation. Neubert ‘05

- Calculation gives constant \( c^{(2)}_0 \) and checks two loop anomalous dim. \( \gamma_1 \) of shape function

\[ c^{(2)}_0 = C_F^2 \left( - \frac{4\pi^2}{3} - \frac{3\pi^4}{40} + 32\zeta_3 \right) + C_F C_A \left( - \frac{326}{81} - \frac{427\pi^2}{108} + \frac{67\pi^4}{180} - \frac{107}{9} \zeta_3 \right) \]
\[ + C_F T_{f n f} \left( - \frac{8}{81} + \frac{5\pi^2}{27} - \frac{20}{9} \zeta_3 \right). \quad \text{TB and Neubert, hep-ph/0512208} \]
Calculation of the jet-function

$$J(p^2) = \frac{1}{\pi} \text{Im}[i \mathcal{J}(p^2)] ; \quad \frac{\eta}{2} \vec{n} \cdot p \mathcal{J}(p^2) = \int d^4x e^{-ipx} \langle 0 | T \left\{ \frac{\eta\bar{\eta}}{4} W^\dagger(0) \psi(0) \bar{\psi}(x) W(x) \frac{\bar{\eta}\eta}{4} \right\} |0\rangle$$

light-cone vectors $n$ and $\bar{n}$

QCD propagator with light-like Wilson lines

$$W(x) = \mathbf{P} \exp \left( ig \int_{-\infty}^0 ds \, \vec{n} \cdot A(x + s\vec{n}) \right)$$

Figure 1: Two-loop diagrams contributing to the jet function in QCD. The circle-cross vertices denote the Wilson lines.
Result for the jet function

- One difficult master integral
  \[ \int d^d k \int d^d l \frac{1}{k^2 l^2 (k + p)^2 (l + p)^2 (k + l + p)^2} \bar{n} \cdot k \bar{n} \cdot l \]

- Use Mellin-Barnes representation, but careful with light-cone propagators

- Check numerically using sector decomposition

- Result for two-loop constant in \( J \)
  \[
  b_0^{(2)} = C_F^2 \left( \frac{205}{8} - \frac{67\pi^2}{6} + \frac{14\pi^4}{15} - 18\zeta_3 \right) + C_F n_f \left( -\frac{4057}{324} + \frac{34\pi^2}{27} + \frac{8}{9}\zeta_3 \right) \\
  + C_A C_F \left( \frac{53129}{648} - \frac{208\pi^2}{27} - \frac{17\pi^4}{180} - \frac{206}{9}\zeta_3 \right) .
  \]
**RG evolution of the jet-function**

\[
\frac{dJ(p^2, \mu)}{d \ln \mu} = - \left[ 2\Gamma_{\text{cusp}}(\alpha_s) \ln \frac{p^2}{\mu^2} + 2\gamma_J(\alpha_s) \right] J(p^2, \mu) \\
- 2\Gamma_{\text{cusp}}(\alpha_s) \int_0^{p^2} dp'^2 \frac{J(p'^2, \mu) - J(p^2, \mu)}{p^2 - p'^2}
\]

**Sudakov factor**

\[
J(p^2, \mu) = \exp \left[ -4S(\mu_i, \mu) + 2a_{\gamma_J} (\mu_i, \mu) \right] \\
\times \tilde{J}(\partial_\eta, \mu_i) \frac{e^{-\gamma E\eta}}{\Gamma(\eta)} \frac{1}{p^2} \left( \frac{p^2}{\mu_i^2} \right)^\eta,
\]

\[
\eta = 2 \int_{\mu_0}^{\mu_i} \frac{d\mu}{\mu} \Gamma_{\text{cusp}}[\alpha_s(\mu)] = 2a_{\Gamma}(\mu_i, \mu).
\]

- Associated jet-function \(\tilde{J}\) is Laplace transform of \(J(p^2, \mu_i)\).
- RG evolution of shape function \(S(\omega, \mu_i)\) has exactly the same form.
“Wonderful formula”

Neubert ’05

\[ F(E_0) = h(m_b, \mu) \int_0^\Delta dP \int_0^P d\omega J(m_b(P - \omega), \mu) S(\omega, \mu) \]

- Plug in solution of RG equations

\[
F(E_0) = \left( \frac{m_b}{\mu_h} \right)^{-2a_r(\mu_h, \mu)} \left( \frac{m_b \Delta}{\mu_i^2} \right)^{2a_r(\mu_i, \mu)} \left( \frac{\Delta}{\mu_0} \right)^{-2a_r(\mu_0, \mu)} \\
\times \exp \left[ 2S(\mu_h, \mu) - 2S(\mu_i, \mu) + 2S(\mu_0, \mu) - 2a_r(\mu_h, \mu_i) - 2a_r(\mu_h, \mu_0) \right] \\
\times h \left( \ln \frac{m_b}{\mu_h}, \mu_h \right) \tilde{j} \left( \ln \frac{m_b \Delta}{\mu_i^2} + \partial_\eta, \mu_h \right) \tilde{s} \left( \ln \frac{\Delta}{\mu_0} + \partial_\eta, \mu_h \right) \frac{e^{-\gamma E \eta}}{\Gamma(1 + \eta)}
\]

\[ \eta = \eta_J + \eta_S = 2a_r(\mu_i, \mu_0) \]

\( h \) is inferred from Melnikov and Mitov ’05
“Wonderful formula”

\[
F(E_0) = \left( \frac{m_b}{\mu_h} \right)^{-2a \Gamma(\mu_h, \mu)} \left( \frac{m_b \Delta}{\mu_i^2} \right)^{2a \Gamma(\mu_i, \mu)} \left( \frac{\Delta}{\mu_0} \right)^{-2a \Gamma(\mu_0, \mu)}
\times \exp \left[ 2S(\mu_h, \mu) - 2S(\mu_i, \mu) + 2S(\mu_0, \mu) - 2a \gamma_j(\mu_h, \mu_i) - 2a \gamma(\mu_h, \mu_0) \right]
\times \hbar \left( \ln \frac{m_b}{\mu_h} + \partial \eta, \mu_h \right) \tilde{j} \left( \ln \frac{m_b \Delta}{\mu_i^2} + \partial \eta, \mu_h \right) \tilde{s} \left( \ln \frac{\Delta}{\mu_0} + \partial \eta, \mu_h \right) \frac{e^{-\gamma E \eta}}{\Gamma(1 + \eta)}
\]

- All scales separated, no large perturbative logarithms.
- Simple analytic expressions for resummed result
  - no need to go to moment space
  - no Landau pole ambiguities
- Very similar expressions for DIS and Drell-Yan in threshold region. → Matthias’ talk
Reduced scale dependence

- Result is very stable under variation of the lowest scale $\mu_0 \sim \Delta \approx 1\,\text{GeV}!$
Summary and Conclusion

• $B \rightarrow X_s \gamma$ is an important constraint on New Physics.

• NNLO calculation is progressing fast
  • Calculation of eff. weak Hamiltonian is almost complete.
  • Matrix elements of $Q_7$ known; approximations for matrix elements of $Q_{1,2}$.
  • Effect of the photon energy cut has been calculated.

• NNLO numbers should follow soon...
extra slides
The need for resummation: fixed order result

\[ \mu = \Delta \Rightarrow \mu = (\Delta m_b)^{1/2} , \mu = m_b ? \]
## Experimental results

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<tr>
<th>Mode</th>
<th>Reported $\mathcal{B}$</th>
<th>$E_{\text{min}}$</th>
<th>$\mathcal{B}$ at $E_{\text{min}}$</th>
<th>Modified $\mathcal{B}$ ($E_{\text{min}} = 1.6$)</th>
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<tr>
<td>CLEO Inc. [264]</td>
<td>$321 \pm 43 \pm 27^{+18}_{-10}$</td>
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<td>$336 \pm 53 \pm 42^{+50}_{-54}$</td>
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