Numerical Multi-Loop Calculations

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Introduction

Perturbative calculations play a key role in understanding and predicting phenomena in particle physics.

- Precise determination of parameters
- Signals of new phenomena
- Complicated backgrounds

Perturbative calculations for processes with higher particle multiplicities, number of loops and kinematical scales

Need to develop and improve tools to deal with real radiation and virtual contributions.
Difficulties and challenges posed by loop integrals

Integrals must be regularized and singularities extracted

Complicated analytic continuation of results when many scales are present

Proliferate the number of terms

Analytic structure (kinematics)

Infrared singularities

Tensor numerators
Difficulties and challenges posed by loop integrals

- Analytic structure (kinematics)
- Infrared singularities
- Tensor numerators

Complicated analytic continuation of results when many scales are present

Integrals must be regularized and singularities extracted

Proliferate the number of terms

...and we would like automatization
Lots of inspired work

- Sophisticated methods to reduce one-loop integrals and handling exceptional kinematics Giele, Glover, R.K. Ellis, Zanderighi, Denner, Dittmaier, Binoth, Heinrich, Pilon, Schubert, Kauer, Hameren, Villinga, Weinzierl, del Aguila, Pittau, Soper, Nagy,…

- “Twistor” developments: on-shell recursion relations Britto, Buchbinder, Cachazo, Feng, Witten, Bern, Dixon, Kosower, Berger, Forde, Mastrolia, …

- Reduction to master integrals and use of differential equations Laporta, Anastasiou, Lazopoulos, A.V. Smirnov, V.A. Smirnov, Tarasov, Baikov, Steinhauser, Gehrmann, Remiddi, Oleari, Tausk,…

- Sector decomposition: numerical evaluation in the euclidean region Binoth, Heinrich, Pilon, Schubert, Kauer

- Mellin-Barnes representations Czakon, V.A. Smirnov, Tausk, Veretin, Anastasiou, Tejeda-Yeomans, Heinrich, Davydychev, Ussyukina,…
Numerical evaluation of loop integrals using Mellin-Barnes representations
Why Mellin-Barnes representations are useful?

- Provide systematic way of extracting infrared singularities.
- Allow to treat tensor integrals avoiding reductions.
- Good numerical convergence (**).
Outline of the method

i. start with a parametric representation of the integral (**)

\[ I = \int \frac{P(x)^\alpha}{Q(x)^\beta} \]

ii. introduce Mellin-Barnes variables to factor polynomials \( P \) and \( Q \) and integrate out parameters

\[ \frac{1}{(a + b)^\alpha} = b^{-\alpha} \frac{1}{2\pi i} \int_C dw \left( \frac{a}{b} \right)^w \frac{\Gamma(-w)\Gamma(\alpha + w)}{\Gamma(\alpha)} \]

iii. analytically continue the integral to make explicit the poles in the regulators

iv. expansion of the resulting integrals

v. integrate numerically the coefficients of the expansion

(**) or use the re-insertion method to get simpler MB representations
A Simple Example: 1 Loop Box

\[ \mathcal{I}_4 = \Gamma (2 + \epsilon) \int \frac{[dx_1 dx_2 dx_3 dx_4]}{(-sx_1 x_3 - tx_2 x_4 - i0)^{2+\epsilon}} \]

\[ \mathcal{I}_4 = \int \left[ \prod_{l=1}^{4} dx_l \right] \frac{1}{2\pi i} \int_C dw \Gamma(-w) \Gamma(2 + \epsilon + w) (-x_2 x_4 t)^w (-x_1 x_3 s)^{-2-\epsilon-w} \]

- integration path must separate poles
- closing contour and summing over residues, reproduces denominator
- Feynman parameters factorize and can be integrated out
A Simple Example: 1 Loop Box

\[
\frac{1}{2\pi i \Gamma(-2\epsilon)} \int_C dw \Gamma(-w) \Gamma(2 + \epsilon + w) \Gamma^2(1 + w) \Gamma^2(-1 - \epsilon - w) (-t)^w (-s)^{-2-\epsilon-w}
\]

the integration path must also separate the new poles
A Simple Example: 1 Loop Box

\[
\frac{1}{2\pi i \Gamma(-2\epsilon)} \int_C dw \Gamma(-w) \Gamma(2 + \epsilon + w) \Gamma^2(1 + w) \Gamma^2(-1 - \epsilon - w) (-t)^w (-s)^{-2-\epsilon-w}
\]

the integration path must also separate the new poles

the representation only holds if \(\epsilon < 0\)
1 Loop Box: analytic continuation

Original representation
$-3/2 < \epsilon < -1/2$

Similar to the original
$-1/2 < \epsilon < 1/2$

Single residue
$-1/2 < \epsilon < 1/2$
$1$ MB variable less
1 Loop Box: analytic continuation

Original representation
\(-\frac{3}{2} < \varepsilon < -\frac{1}{2}\)

Similar to the original
\(-\frac{1}{2} < \varepsilon < \frac{1}{2}\)

Single residue
\(-\frac{1}{2} < \varepsilon < \frac{1}{2}\)
1 MB variable less

\[- \frac{\varepsilon}{\pi i} \int_C dw \Gamma(-w) \Gamma(2 + w) \Gamma^2(1 + w) \Gamma^2(-1 - w) (-t)^w (-s)^{-2-w} + \mathcal{O}(\varepsilon^2)\]

\[+ \frac{(-t)^{-\varepsilon}}{s t} \frac{\Gamma(-\varepsilon)^2 \Gamma(1 + \varepsilon)}{\Gamma(-2\varepsilon)} \left\{ \gamma_E - \log(-s) + \log(-t) + 2\psi(-\varepsilon) - \psi(1 + e) \right\} \]
1 Loop Box: analytic continuation

- Iterative procedure, until reaching $\epsilon = 0$.
- Poles in $\epsilon$ appear explicitly when taking residues.
- Remaining integrals can be expanded in Laurent series.
- Straightforward to extend to more complicated loop integrals.
- Calculation of remaining integrals:
  - Analytically, series resummation
  - Well suited for numerical integration (**)
- Invariants appear in simple powers and logarithms:
  trivial analytic continuation for numerical evaluation

\[
- \frac{\epsilon}{\pi i} \int_C dw \Gamma(-w)\Gamma(2+w)\Gamma^2(1+w)\Gamma^2(-1-w)(-t)^w (-s)^{-2-w} + \mathcal{O}(\epsilon^2) \\
+ \frac{(-t)^{-\epsilon}}{st} \frac{\Gamma(-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(-2\epsilon)} \left\{ \gamma_E - \log(-s) + \log(-t) + 2\psi(-\epsilon) - \psi(1+\epsilon) \right\}
\]
Numerical convergence

- gamma functions coming from integrating Feynman parameter are crucial to improve numerical convergence
- invariants appear attached to two Feynman variables:
  - additional gamma functions in numerator
  - cancellation of MB variables in the denominators
  - integrand vanishes fastly when moving away from the real line
- terms with internal masses are linear in Feynman parameters:
  - missing gamma functions in numerator
  - extra MB variables in the denominators
  - lose of damping, oscillatory integrands
Tensor integrals

\[ I_{n,m} = \int \frac{d^d k_i}{i \pi^{\frac{d}{2}}} \frac{k_{\mu_1} \ldots k_{\mu_m}}{(k + q_1)^2 (k + q_2)^2 \cdots (k + q_n)^2} \]

\[ = \sum_{r \leq m} \int \left( \prod_{i} dw_i \right) \Gamma^{(\text{scalar})}_{d \rightarrow d+r} (\vec{w}) h^{(m,r)} (\vec{w}) \]

\( \Gamma^{(\text{scalar})}_{d \rightarrow d+r} (\vec{w}) \): analogous to the representation of the scalar case with shifted dimension \( d \rightarrow d + r \)

\( h^{(m,r)} (\vec{w}) \): polynomial in the MB variables with tensor coefficients (external momenta). Does not affect the analytic continuation in \( \epsilon \)

-This decomposition allows to perform a unique analytic continuation in \( \epsilon \) for each topology, using a general polynomial.
-Whole diagrams can be evaluated at the same time.
Results

Selection of results
- checks with known integrals in different kinematical regions
- evaluation of tensor integrals
- numerical calculation of master integrals

Recent related work
- similar public program by Czakon (hep-ph/0511200)
- impressive application of the method to prove the iteration of 5-point amplitudes at 2 loops in N4SYM (Z. Bern et al., hep-th/0604074)
1 loop massless hexagon

- many important processes with six external legs at the LHC
- first result for rank 6 tensors for the hexagon topology
- recent calculations of 6 points amplitudes
  6 gluons, semi-numerical (talk by W. Giele) [Ellis, Giele, Zanderighi]
  on-shell recursions (talks by C. Berger and L. Dixon) [Berger, Bern, Del Duca, Dixon, Dunbar, Forde, Kosower]
- our calculation:
  8 dimensional MB representation
  analytic continuation for arbitrary tensors
  sampled several points in the physical region $2 \rightarrow 4$
  check in the euclidean region with sector decomposition
  check some phase space points with explicit reductions using IBP identities
- ready to evaluate whole diagrams
- so far not competitive with semi-numerical methods
2 loop boxes

- planar double box with one off-shell leg
  test of analytic continuation in $\epsilon$
  effective 3-fold integral
  analytical results by V.A. Smirnov (MB), Gehrmann and Remiddi (diff. eq.)
  non-trivial check of analytic continuation

- planar double box with two adjacent off-shell legs
  relevant for calculation of heavy boson pair production at colliders
  first calculation in the physical region
  euclidean points calculated by Binoth and Heinrich with sec. dec.
  independent check in the euclidean region with sec. dec.
  effective 5-fold integral
2-Box: 1 off-shell leg

\[ M^2 = 1 \quad s_{13} = \frac{3}{10} \quad s_{23} \]

\[ M^2 = \frac{1}{10} \quad s = 1 \]
2-Box: 2 off-shell legs

\[ s = 1 \quad M_1^2 = \frac{1}{20} \quad M_2^2 = \frac{1}{2} \]
3 loop boxes

- on-shell triple box
  analytic calculation by V.A. Smirnov with MB technique
  evaluation in different physical regions
  poles up to $e^{-6}$
  effective 5-fold integral

- triple box with one off-shell leg
  first evaluation of a 3 loop box with 3 scales
  evaluation in different physical regions
  numerical computation of 8 dimensional MB integrals
  production of a heavy boson in association with a jet
  effective 6-fold integral
3-Box: on-shell
3-Box: 1 off-shell leg

\[ M^2 = 1 \quad s_{13} = 3/10 \]

\[ M^2 = 1/10 \quad s = 1 \]
Summary

- framework for the numerical evaluation of loop integrals using Mellin-Barnes representations
  - algorithmic extraction of infrared singularities
  - very well suited for multi-loop multi-scale problems
  - deals with tensor integrals
  - direct numeric integration of contour integrals
  - full automatization of the whole procedure

- to do:
  - complete physical application
  - make code public
  - study massive cases

- promising results showing the strength of the framework:
  - master integrals:
    - two loop scalar box with two external masses
    - three loop scalar box with one external mass
  - tensor integrals:
    - rank six tensors for the 6 point amplitude at one loop