Numerical Multi-Loop Calculations

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Introduction

Perturbative calculations play a key role in understanding and predicting phenomena in particle physics

Precise determination of parameters

Signals of new phenomena

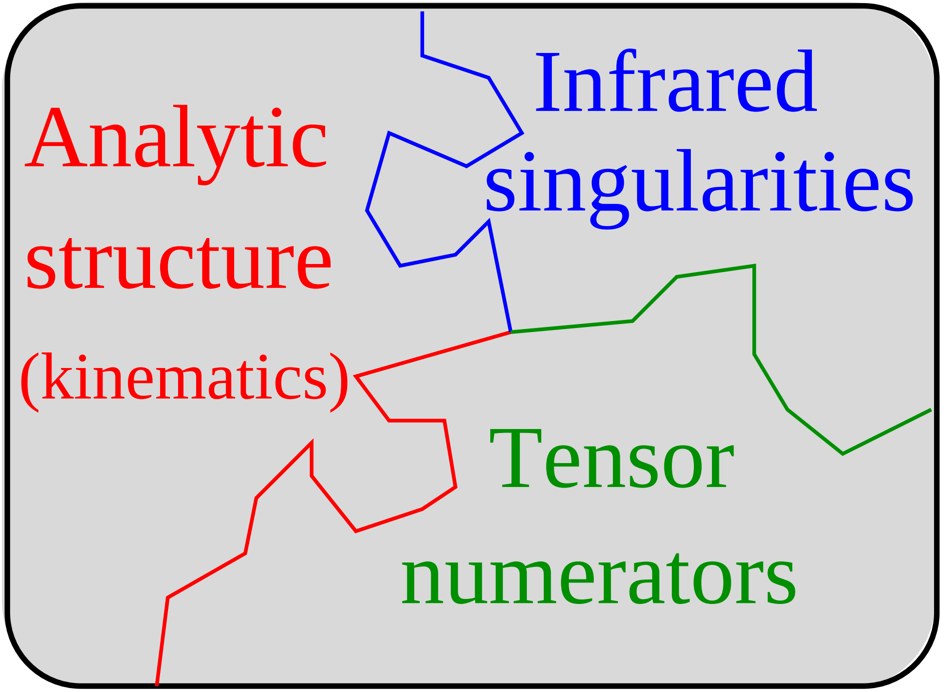
Complicated backgrounds

Perturbative calculations for processes with higher particle multiplicities, number of loops and kinematical scales

Need to develop and improve tools to deal with real radiation and virtual contributions

Difficulties and challenges posed by loop integrals

Complicated analytic continuation of results when many scales are present

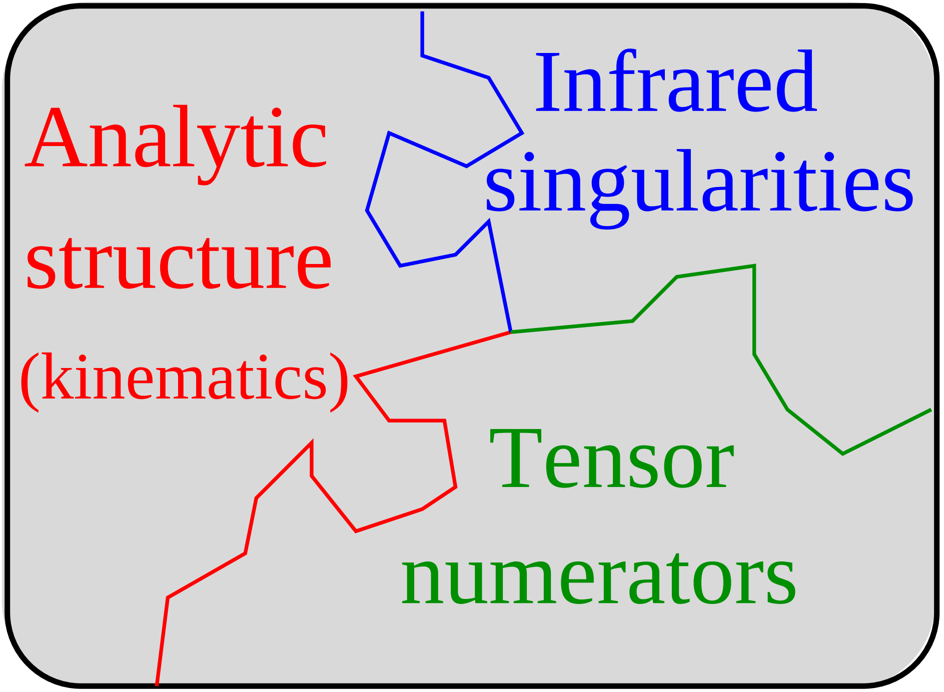


Integrals must be regularized and singularities extracted

Proliferate the number of terms

Difficulties and challenges posed by loop integrals

Complicated analytic continuation of results when many scales are present



Integrals must be regularized and singularities extracted

Proliferate the number of terms

...and we would like automatization

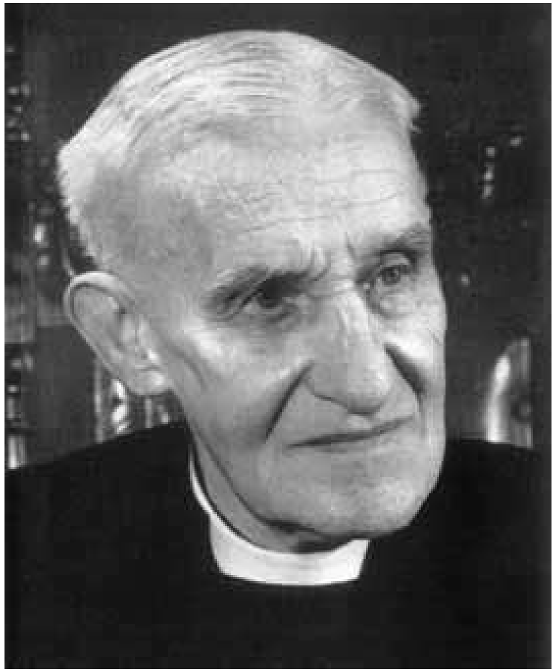
Lots of inspired work

- *Sophisticated methods to reduce one-loop integrals and handling exceptional kinematics* *Giele, Glover, R.K. Ellis, Zanderighi, Denner, Dittmaier, Binoth, Heinrich,Pilon, Schubert, Kauer, Hameren, Vollinga, Weinzierl, del Aguila, Pittau, Soper, Nagy,. . .*
- *"Twistor" developments: on-shell recursion relations* *Britto, Buchbinder, Cachazo, Feng, Witten, Bern, Dixon, Kosower, Berger, Forde, Mastrolia, . . .*
- *Reduction to master integrals and use of differential equations* *Laporta, Anastasiou, Lazopoulos, A.V. Smirnov, V.A. Smirnov, Tarasov, Baikov, Steinhauser, Gehrmann, Remiddi, Oleari, Tausk,. . .*
- *Sector decomposition: numerical evaluation in the euclidean region* *Binoth, Heinrich, Pilon, Schubert, Kauer*
- *Mellin-Barnes representations* *Czakon, V.A. Smirnov, Tausk, Veretin, Anastasiou, Tejeda-Yeomans, Heinrich, Davydychev, Ussyukina,. . .*

Numerical evaluation of loop integrals using Mellin-Barnes representations

Why Mellin-Barnes representations are useful?

- *Provide systematic way of extracting infrared singularities.*
- *Allow to treat tensor integrals avoiding reductions.*
- *Good numerical convergence (**).*



Outline of the method

i. *start with a parametric representation of the integral (**)*

$$\mathcal{I} = \int [dx] \, \frac{\mathcal{P}(x)^\alpha}{\mathcal{Q}(x)^\beta}$$

ii. *introduce Mellin-Barnes variables to factor polynomials $\mathcal{P}$ and $\mathcal{Q}$ and integrate out parameters*

$$\frac{1}{(a+b)^\alpha} = b^{-\alpha} \frac{1}{2\pi i} \int_{\mathcal{C}} dw \left(\frac{a}{b}\right)^w \frac{\Gamma(-w)\,\Gamma(\alpha+w)}{\Gamma(\alpha)}$$

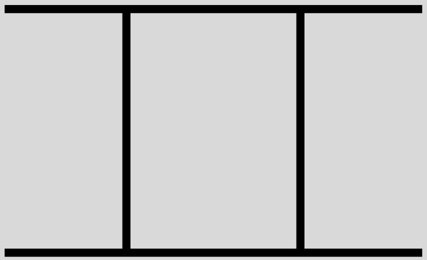
iii. *analytically continue the integral to make explicit the poles in the regulators*

iv. *expansion of the resulting integrals*

v. *integrate numerically the coeffi cients of the expansion*

(**) or use the re-insertion method to get simpler MB representations

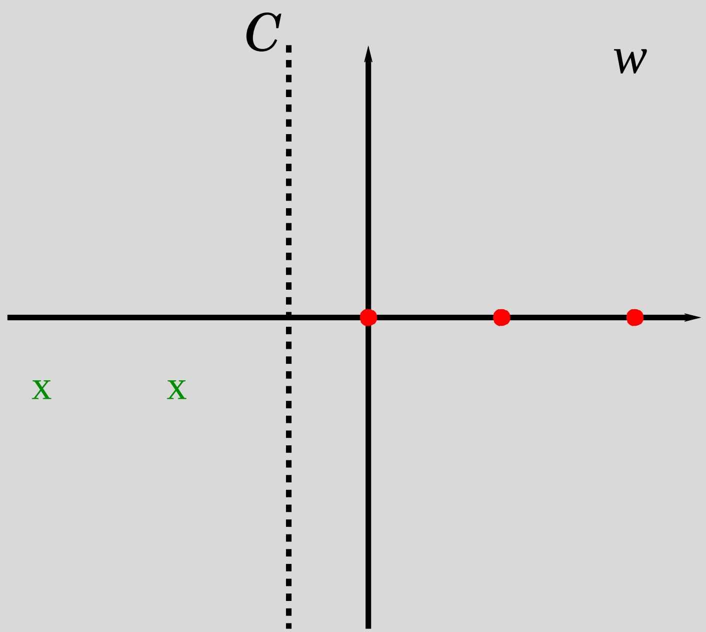
A Simple Example: 1 Loop Box



$$\mathcal{I}_4 = \Gamma\left(2+\epsilon\right) \int \frac{\left[dx_1 dx_2 dx_3 dx_4\right]}{\left(-s x_1 x_3 - t x_2 x_4 - i0\right)^{2+\epsilon}}$$

$$\mathcal{I}_4 = \int \left[\prod_{l=1}^{4} dx_l\right] \frac{1}{2\pi i} \int_{\mathcal{C}} dw \Gamma(-w) \Gamma(2+\epsilon+w) \ \left(-x_2 x_4 t\right)^{w} \left(-x_1 x_3 s\right)^{-2-\epsilon-w}$$

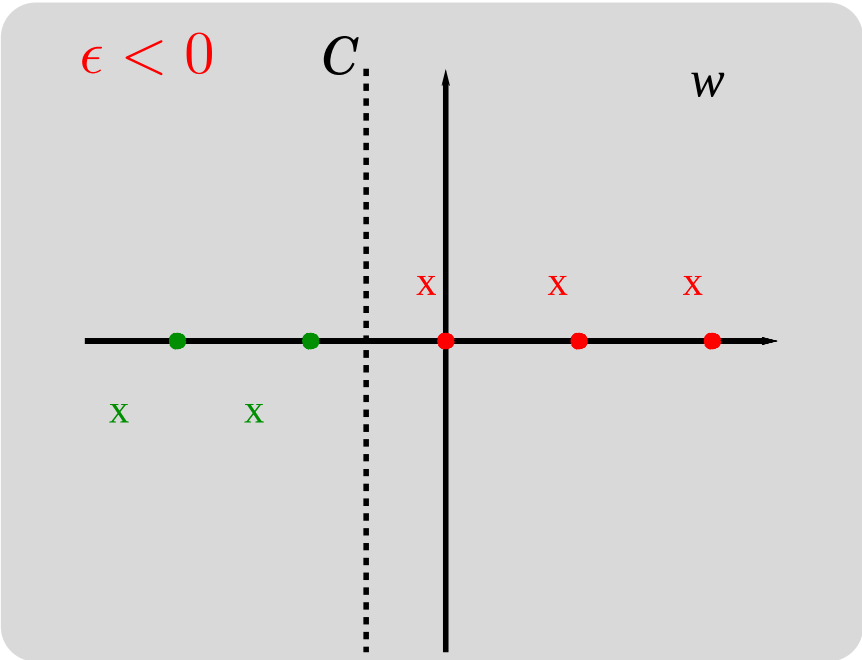
- *integration path must separate poles*
- *closing contour and summing over residues, reproduces denominator*
- *Feynman parameters factorize and can be integrated out*



A Simple Example: 1 Loop Box

$$\frac{1}{2\pi i\,\Gamma(-2\epsilon)}\int_{\mathcal{C}} dw\,\Gamma(-w)\Gamma(2+\epsilon+w)\Gamma^2(1+w)\Gamma^2(-1-\epsilon-w)\,(-t)^w\,(-s)^{-2-\epsilon-w}$$

the integration path must also separate the new poles



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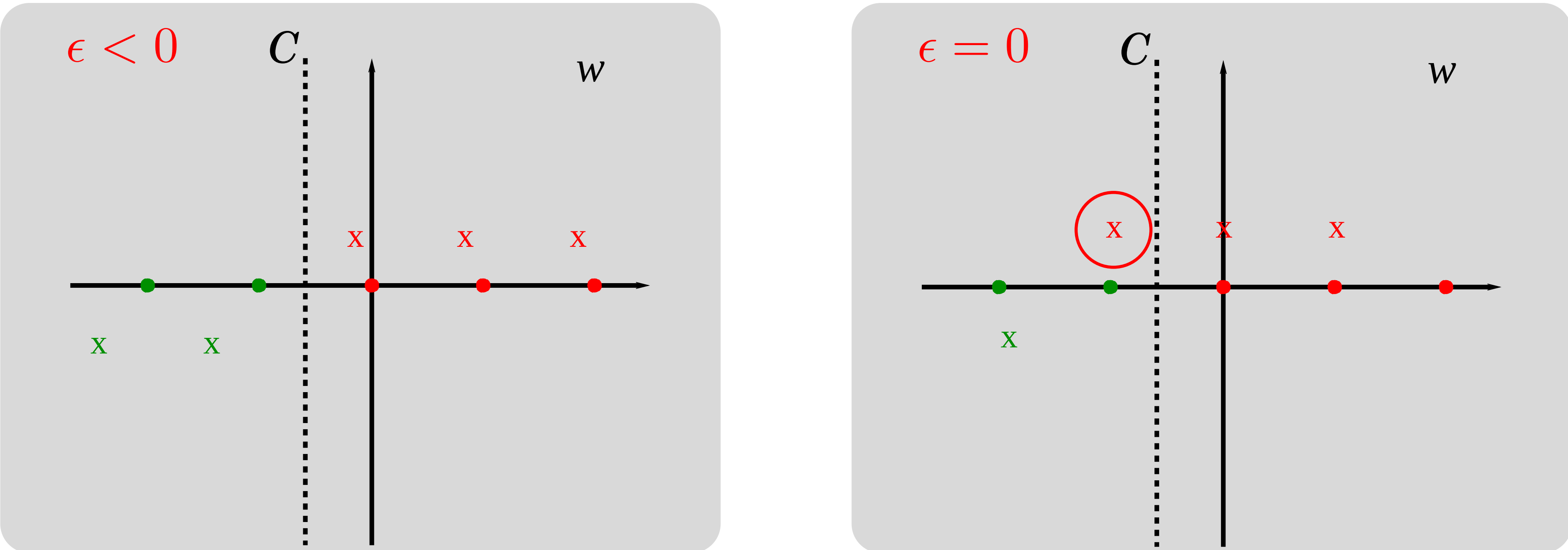
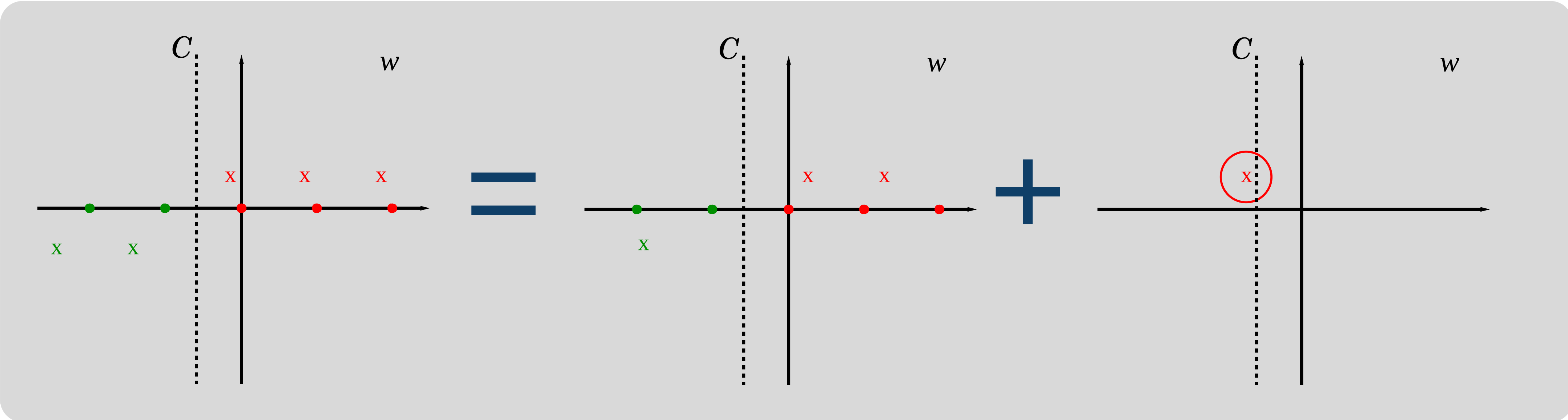
$$\frac{1}{2\pi i\,\Gamma(-2\epsilon)}\int_{\mathcal{C}} dw\,\Gamma(-w)\Gamma(2+\epsilon+w)\Gamma^2(1+w)\Gamma^2(-1-\epsilon-w)\,(-t)^w\,(-s)^{-2-\epsilon-w}$$

the integration path must also separate the new poles

the representation only holds if $\epsilon < 0$



1 Loop Box: analytic continuation

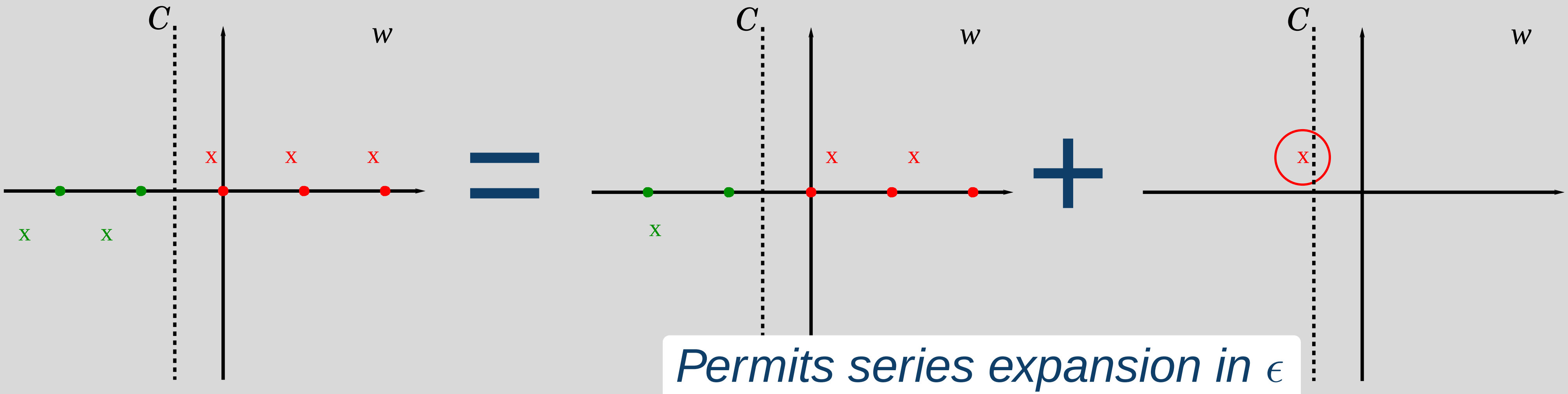


Original representation
$-3/2 < \epsilon < -1/2$

Similar to the original
$-1/2 < \epsilon < 1/2$

Single residue
$-1/2 < \epsilon < 1/2$
1 MB variable less

1 Loop Box: analytic continuation



Permits series expansion in ϵ

Original representation
$-3/2 < \epsilon < -1/2$

Similar to the original
$-1/2 < \epsilon < 1/2$

Single residue
$-1/2 < \epsilon < 1/2$
1 MB variable less

$$- \quad \frac{\epsilon}{\pi i} \int_{\mathcal{C}} dw \Gamma(-w) \Gamma(2+w) \Gamma^2(1+w) \Gamma^2(-1-w) \, (-t)^w \, (-s)^{-2-w} + \mathcal{O}(\epsilon^2)$$

$$+ \quad \frac{(-t)^{-\epsilon}}{s\,t} \frac{\Gamma(-\epsilon)^2 \Gamma(1+\epsilon)}{\Gamma(-2\epsilon)} \left\{ \gamma_E - \log(-s) + \log(-t) + 2\psi(-e) - \psi(1+e) \right\}$$

1 Loop Box: analytic continuation

- *Iterative procedure, until reaching $\epsilon = 0$.*
- *Poles in ϵ appear explicitly when taking residues.*
- *Remaining integrals can be expanded in Laurent series.*
- *Straightforward to extend to more complicated loop integrals.*
- *Calculation of remaining integrals:*
  $\star$ *Analytically, series resummation*
  $\star$ *Well suited for numerical integration (**)*
- *Invariants appear in simple powers and logarithms:*
  trivial analytic continuation for numerical evaluation

$$
\begin{aligned}
- \quad & \frac{\epsilon}{\pi i}\int_{\mathcal{C}} dw\,\Gamma(-w)\Gamma(2+w)\Gamma^2(1+w)\Gamma^2(-1-w)\,(-t)^{w}\,(-s)^{-2-w} + \mathcal{O}(\epsilon^2) \\
+ \quad & \frac{(-t)^{-\epsilon}}{s\,t}\frac{\Gamma(-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(-2\epsilon)}\left\{\gamma_E - \log(-s) + \log(-t) + 2\psi(-e) - \psi(1+e)\right\}
\end{aligned}
$$

Numerical convergence

- *gamma functions coming from integrating Feynman parameter are crucial to improve numerical convergence*
- *invariants appear attached to two Feynman variables:*
  - *additional gamma functions in numerator*
  - *cancellation of MB variables in the denominators*
  - *integrand vanishes fastly when moving away from the real line*
- *terms with internal masses are linear in Feynman parameters:*
  - *missing gamma functions in numerator*
  - *extra MB variables in the denominators*
  - *lose of damping, oscillatory integrands*

Tensor integrals

$$I_{n,m} = \int \frac{d^d k}{i\,\pi^{\frac{d}{2}}} \frac{k^{\mu_1}\dots k^{\mu_m}}{(k+q_1)^2(k+q_2)^2\cdots(k+q_n)^2}$$

$$= \sum_{r\le m} \int \left(\prod_i dw_i\right) \Gamma^{(scalar)}_{d\to d+r}(\vec{w})\, h^{(m,r)}(\vec{w})$$

$\Gamma^{(scalar)}_{d\to d+r}(\vec{w})$: *analogous to the representation of the scalar case with shifted dimension* $d \to d+r$

$h^{(m,r)}(\vec{w})$: *polynomial in the MB variables with tensor coefficients (external momenta). Does not affect the analytic continuation in* ϵ

-This decomposition allows to perform a unique analytic continuation in ϵ *for each topology, using a general polynomial.*
-Whole diagrams can be evaluated at the same time.

Results

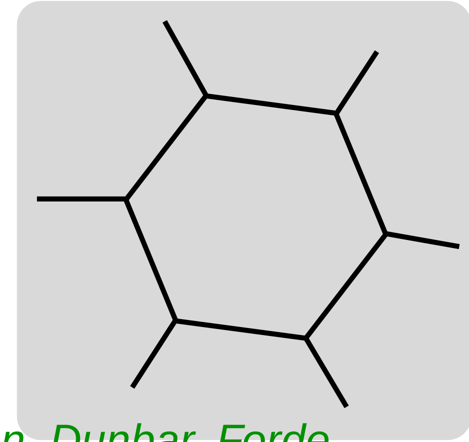
Selection of results

- *checks with known integrals in different kinematical regions*
- *evaluation of tensor integrals*
- *numerical calculation of master integrals*

Recent related work

- *similar public program by Czakon (hep-ph/0511200)*
- *impressive application of the method to prove the iteration of 5-point amplitudes at 2 loops in N4SYM (Z. Bern et al., hep-th/0604074)*

1 loop massless hexagon



- many important processes with six external legs at the LHC
- first result for rank 6 tensors for the hexagon topology
- recent calculations of 6 points amplitudes

  6 gluons, semi-numerical (talk by W. Giele) [Ellis, Giele, Zanderighi]

  on-shell recursions (talks by C. Berger and L. Dixon) [Berger, Bern, Del Duca, Dixon, Dunbar, Forde, Kosower]

- our calculation:

  8 dimensional MB representation
  analytic continuation for arbitrary tensors
  sampled several points in the physical region $2 \rightarrow 4$
  check in the euclidean region with sector decomposition
  check some phase space points with explicit reductions using IBP identities

- ready to evaluate whole diagrams
- so far not competitive with semi-numerical methods

2 loop boxes

- *planar double box with one off-shell leg*

  test of analytic continuation in ϵ

  effective 3-fold integral

  analytical results by V.A. Smirnov (MB), Gehrmann and Remiddi (diff. eq.)
  non-trivial check of analytic continuation

- *planar double box with two adjacent off-shell legs*

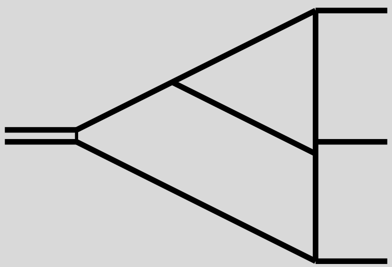
  relevant for calculation of heavy boson pair production at colliders

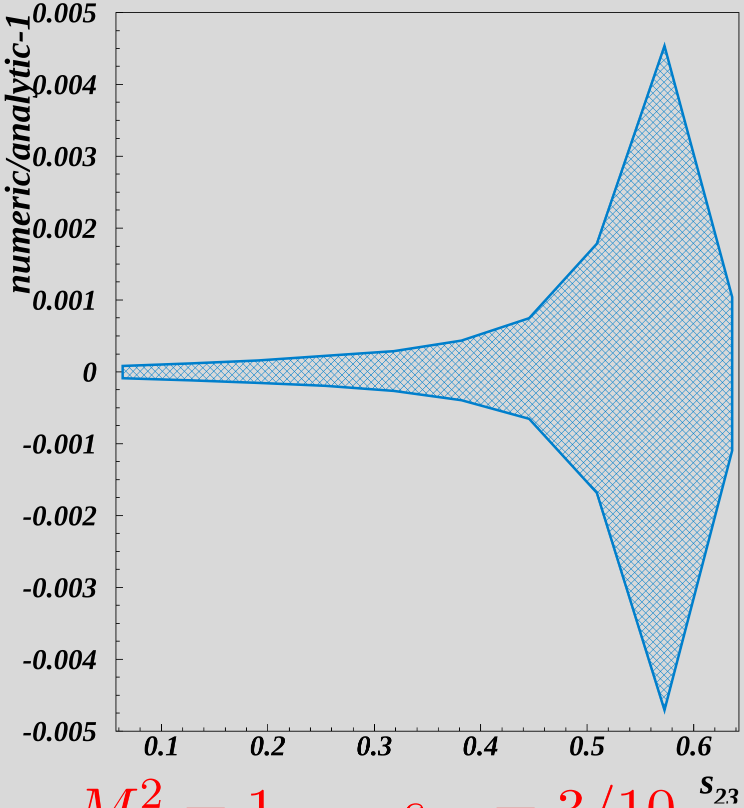
  fi rst calculation in the physical region

  euclidean points calculated by Binoth and Heinrich with sec. dec.

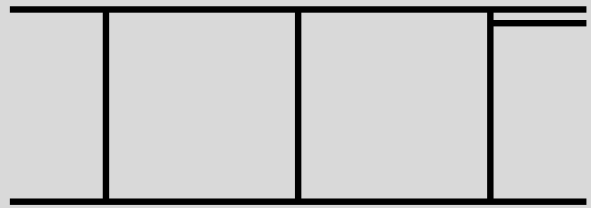
  independent check in the euclidean region with sec. dec.
  effective 5-fold integral

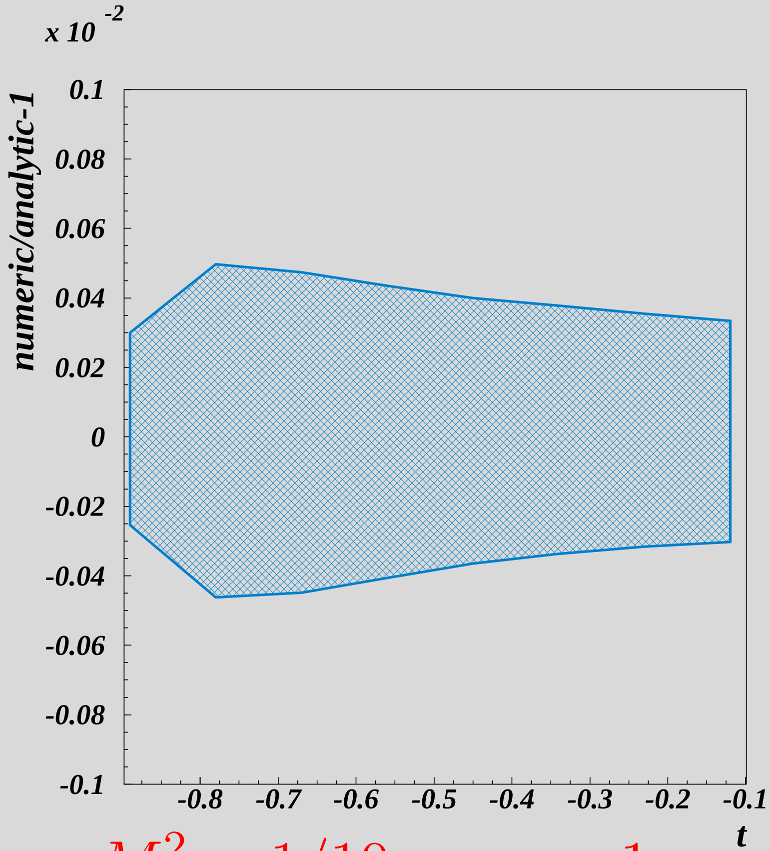
2-Box: 1 off-shell leg





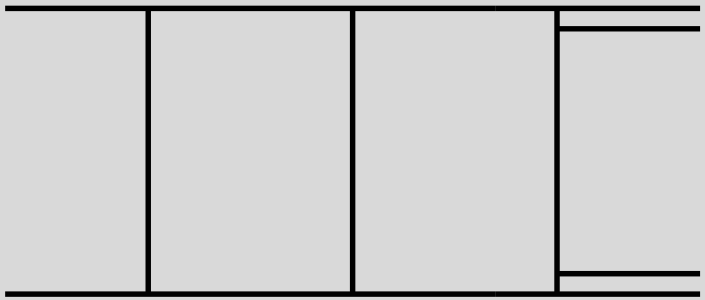
$M^2 = 1$ $\quad$ $s_{13} = 3/10$

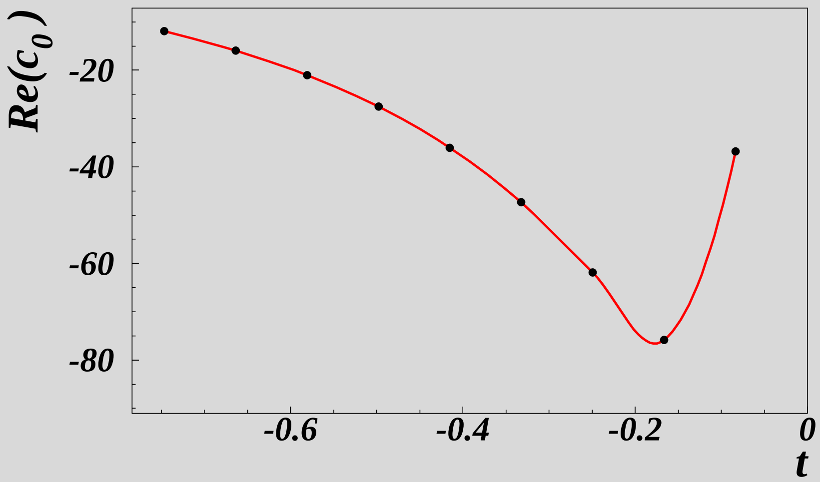


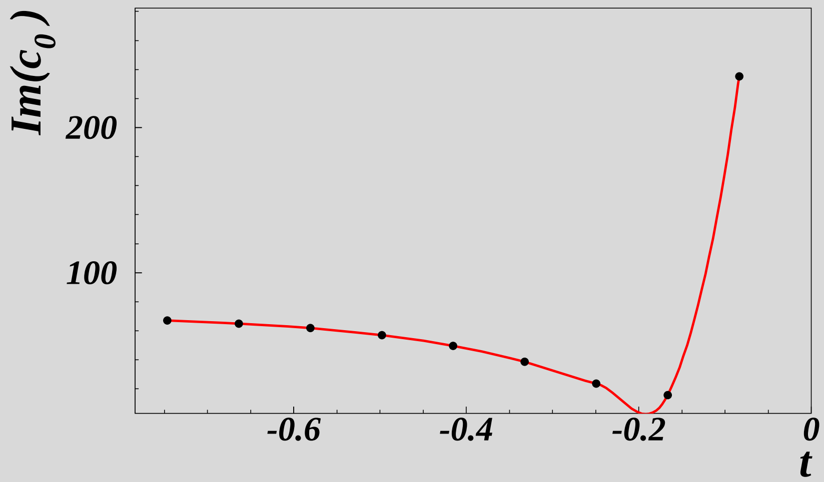


$M^2 = 1/10$ $\quad$ $s = 1$

2-Box: 2 off-shell legs







$s = 1 \qquad M_1^2 = 1/20 \qquad M_2^2 = 1/2$

3 loop boxes

- *on-shell triple box*

  analytic calculation by V.A. Smirnov with MB technique

  evaluation in different physical regions

  poles up to e^{-6}

  effective 5-fold integral

- *triple box with one off-shell leg*

  fi rst evaluation of a 3 loop box with 3 scales

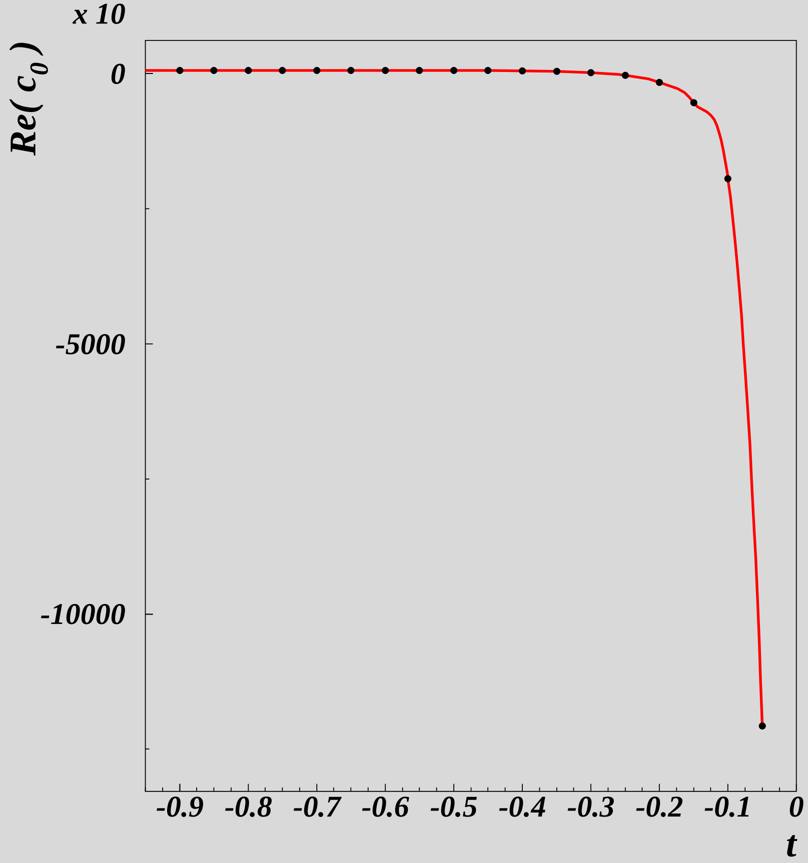
  evaluation in different physical regions

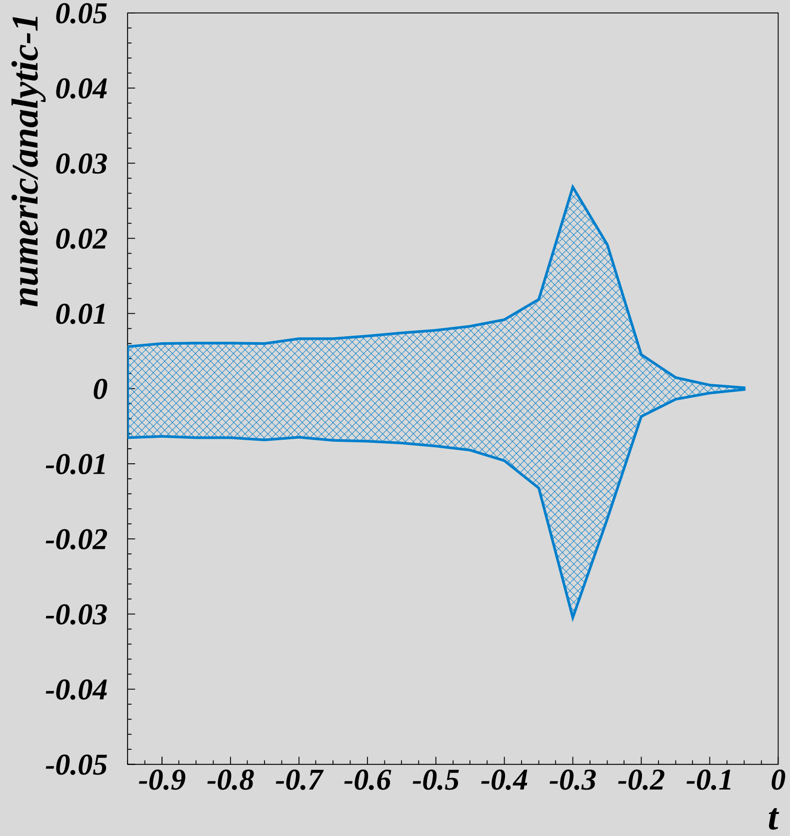
  numerical computation of 8 dimensional MB integrals

  production of a heavy boson in association with a jet

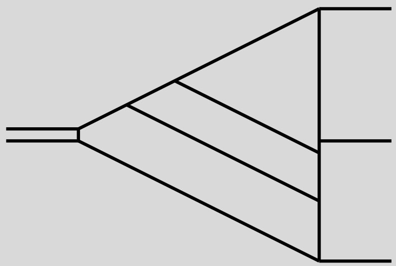
  effective 6-fold integral

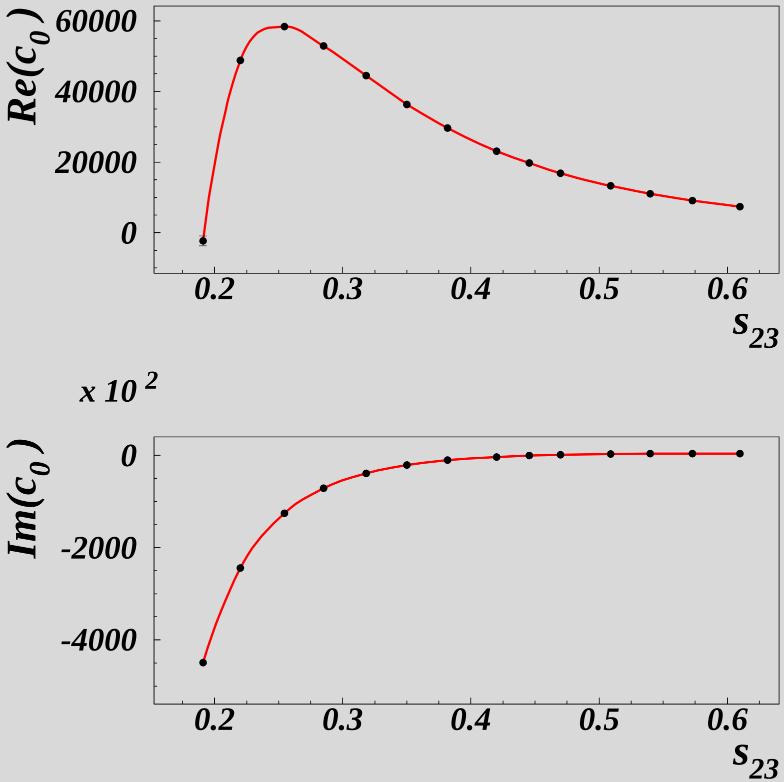
3-Box: on-shell



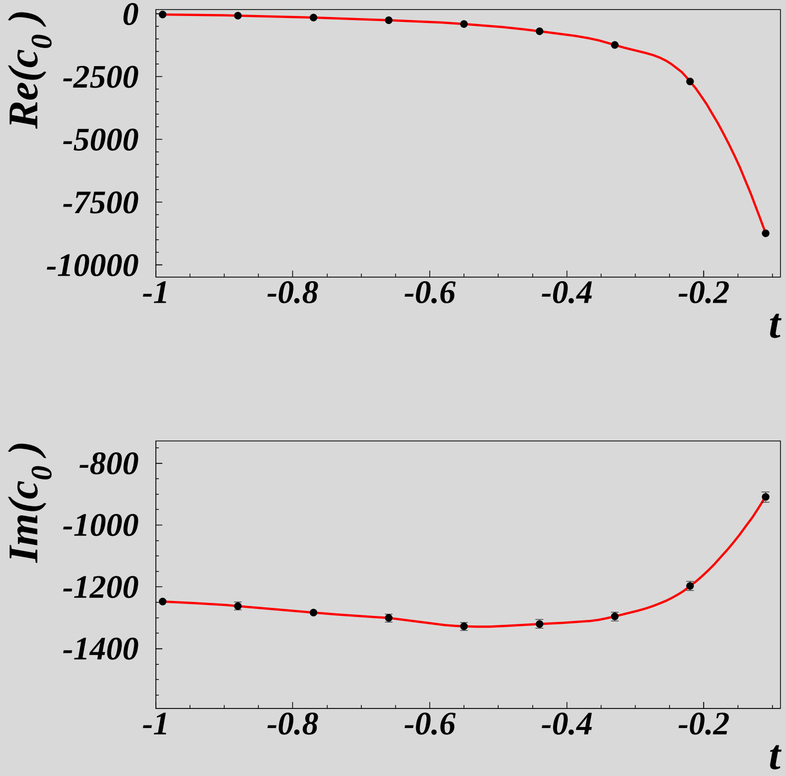


3-Box: 1 off-shell leg





$M^2 = 1 \qquad s_{13} = 3/10$



$M^2 = 1/10 \qquad s = 1$

Summary

- *framework for the numerical evaluation of loop integrals using Mellin-Barnes representations*

  algorithmic extraction of infrared singularities
  very well suited for multi-loop multi-scale problems
  deals with tensor integrals
  direct numeric integration of contour integrals
  full automatization of the whole procedure

- *to do:*

  complete physical application
  make code public
  study massive cases

- *promising results showing the strength of the framework:*

  $\star$ *master integrals:*

  two loop scalar box with two external masses
  three loop scalar box with one external mass

  $\star$ *tensor integrals:*

  rank six tensors for the 6 point amplitude at one loop