Multi-Loop Miracles in N=4 Super-Yang-Mills Theory

- Z. Bern, L.D., V. Smirnov, hep-th/0505205
- F. Cachazo, M. Spradlin, A. Volovich, hep-th/0602228
- Z. Bern, M. Czakon, D. Kosower, R. Roiban, V. Smirnov, hep-th/0604074

LoopFest V, SLAC
June 19, 2006
Outline

• N=4 SYM and AdS/CFT
• Use of unitarity to construct N=4 SYM amplitude integrands
• Exponentiated infrared structure of amplitudes and Sudakov form factors
• Exponentiation of full amplitudes (including IR finite terms) in planar N=4 SYM at 2 and 3 loops
• “Leading transcendentality” connection between terms in QCD and N=4 SYM
• Conclusions
N=4 SYM, AdS/CFT
and perturbative scattering

- **N=4 SYM** is most supersymmetric theory possible without gravity
- Uniquely specified by gauge group, say $SU(N_c)$
- **Exactly scale-invariant** (conformal) field theory: $\beta(g) = 0$
- AdS/CFT duality suggests that **weak-coupling** perturbation series for planar (large $N_c$) N=4 SYM should have special properties:
  - strong-coupling limit equivalent to weakly-coupled gravity theory
- Some quantities are protected, unrenormalized, so the series is **trivial** (e.g. energies of BPS states)
- **Scattering amplitudes** (near $D=4$) are not protected – how does series organize itself into a simple result, from gravity point of view?
N=4 SYM particle content

- 1 massless spin 1 gluon
- 4 massless spin 1/2 gluinos
- 6 massless spin 0 scalars

SUSY
$Q_a$, $a=1,2,3,4$
shifts helicity by 1/2

$\mathcal{N} = 4$
1 $\leftrightarrow$ 4 $\leftrightarrow$ 6 $\leftrightarrow$ 4 $\leftrightarrow$ 1

$g^-$, $\lambda^-_i$, $\phi_{ij}$, $\phi_{ij}$, $\lambda^+_i$, $g^+$

helicity
-1 $\rightarrow$ $-\frac{1}{2}$ $\rightarrow$ 0 $\rightarrow$ $\frac{1}{2}$ $\rightarrow$ 1

all in adjoint representation
N=4 SYM interactions

All proportional to same *dimensionless* coupling constant, $g$

- **SUSY cancellations:** scale invariance *preserved* quantum mechanically

$$\frac{d\alpha}{d \ln \mu^2} = \beta(\alpha) = \left[ 6 \times \frac{1}{6} + 4 \times \frac{2}{3} - \frac{11}{3} \right] \frac{N_c \alpha^2}{4\pi} = 0 \quad \text{(true to all orders in $\alpha$)}$$

June 19, 2006

L. Dixon

Multi-loop Miracles in Planar N=4 SYM
Perturbative Unitarity

• \( S \)-matrix is a unitary operator between in and out states

\[
1 = S^\dagger S = (1 - iT^\dagger)(1 + iT)
\]

\[
2 \text{Im} T = T^\dagger T
\]

• Expand \( T \)-matrix in \( g \)

\[
T_4 = g^2 + g^4 + g^6 + \cdots
\]

\[
T_5 = g^3 + g^5 + \cdots
\]

• Very efficient due to simple structure of tree helicity amplitudes

Bern, LD, Dunbar, Kosower (1994)
Many higher-loop contributions to $gg \rightarrow gg$ scattering can be deduced from a simple property of the 2-particle cuts at one loop (Bern, Rozowsky, Yan 1997).

Leads to “rung rule” for easily computing all contributions which can be built by iterating 2-particle cuts.
“Iterated 2-particle cut-constructible contributions”

For example, the coefficient of this topology is easily computable from the rung rule

More concise terminology: (planar) Mondrian diagrams
Simplicity of N=4 SYM 4-point amplitudes

• 1 loop:

\[ i s_{12} s_{23} \left[ \begin{array}{c}
\text{planar} \\
\end{array} \right] \]

where

\[ \int \frac{d^{d-2} \ell_1}{(2\pi)^{d-2}} \frac{1}{\ell_1^2 (\ell_1 - k_1)^2 (\ell_1 - k_1 - k_2)^2 (\ell_1 + k_4)^2} \]

\[ = \delta^{ab} \]

“color dresses kinematics”

• 2 loops:

\[ i^2 s_{12} s_{23} \left[ \begin{array}{c}
\text{planar} \\
\end{array} \right] \]

Bern, Rozowsky, Yan (1997); Bern, LD, Dunbar, Perelstein, Rozowsky (1998)

• Analogous computation in QCD not completed until 2001

Glover, Oleari, Tejeda-Yeomans (2001); Bern, De Freitas, LD (2002)
Three loop planar amplitude

- 3-loop planar diagrams (leading terms for large $N_c$):
  
  $$ i^3 s_{12} s_{23} \left[ s_{12}^2 + s_{23}^2 \right] $$
  
  $$ + 2 s_{12}(l+k_4)^2 \left[ 2 s_{12}^2(l+k_1)^2 \right] $$

  BRY (1997); BDDPR (1998)

  (3-particle cuts also checked)

- 3-loop non-planar result
  includes non-Mondrian diagrams
  – not completed yet
IR Structure of QCD and N=4 SYM Amplitudes

- Expand multi-loop amplitudes in $d=4-2\epsilon$ around $d=4$ ($\epsilon=0$)

- Overlapping soft + collinear divergences at each loop order imply leading poles are $\sim 1/\epsilon^{2L}$ at $L$ loops

- Pole terms are predictable, up to constants, for QCD & N=4 SYM, due to soft/collinear factorization and exponentiation

Mueller (1979); Collins (1980); Sen (1981); Sterman (1987)
Catani, Trentadue (1989); Korchemsky (1989)
Magnea, Sterman (1990); Korchemsky, Marchesini, hep-ph/9210281

- Surprise is that, for planar N=4 SYM (only), the finite ($\epsilon^0$) terms also exponentiate!
Soft/Collinear Factorization

\[ M = S(k_i, \mu, \alpha_s(\mu), \epsilon) \times \prod_{i=1}^{n} J_i(\mu, \alpha_s(\mu), \epsilon) \times h_n(k_i, \mu, \alpha_s(\mu), \epsilon) \]

- **S** = soft function (only depends on color of \(i^{th}\) particle)
- **J** = jet function (color-diagonal; depends on \(i^{th}\) spin)
- **\(h_n\)** = hard remainder function (finite as \(\epsilon \rightarrow 0\))
Simplification at Large $N_c$ (Planar Case)

- **Soft function** only defined up to a multiple of the identity matrix in color space
- Planar limit is color-trivial; can absorb $S$ into $J_i$
- If all $n$ particles are identical, say gluons, then each “wedge” is the square root of the “gg -> 1” process (Sudakov form factor):

\[
M_n = \prod_{i=1}^{n} \left[ M_{gg \to 1} \left( \frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon \right) \right]^{1/2} \times h_n (k_i, \mu, \alpha_s, \epsilon)
\]
Sudakov form factor

By analyzing structure of soft/collinear terms in axial gauge, find differential equation for form factor $M^{[gg \rightarrow 1]}(Q^2/\mu^2, \alpha_s(\mu), \epsilon)$:

$$\frac{\partial}{\partial \ln Q^2} M^{[gg \rightarrow 1]}(Q^2/\mu^2, \alpha_s(\mu), \epsilon) = \frac{1}{2} \left[ K(\epsilon, \alpha_s) + G(Q^2/\mu^2, \alpha_s(\mu), \epsilon) \right] \times M^{[gg \rightarrow 1]}(Q^2/\mu^2, \alpha_s(\mu), \epsilon)$$

finite as $\epsilon \rightarrow 0$; contains all $Q^2$ dependence

Pure counterterm (series of $1/\epsilon$ poles); like $\beta(\epsilon, \alpha_s)$, single poles in $\epsilon$ determine $K$ completely

$K, G$ also obey differential equations (ren. group):

$$\left( \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \alpha_s} \right) (K + G) = 0$$

$$\left( \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \alpha_s} \right) K = -\gamma_K(\alpha_s)$$

soft or cusp anomalous dimension
Sudakov form factor (cont.)

• Solution to differential equations for $\mathcal{M}^{[gg\rightarrow 1]}$, $G$:

\[
\mathcal{M}^{[gg\rightarrow 1]}(Q^2/\mu^2, \alpha_s(\mu), \epsilon) = \exp\left\{ \frac{1}{2} \int_0^{-Q^2} \frac{d\xi^2}{\xi^2} [K(\epsilon, \alpha_s(\mu)) + G(-1, \bar{\alpha}_s(\mu^2/\xi^2, \alpha_s(\mu), \epsilon), \epsilon) \right. \\
\left. + \frac{1}{2} \int_0^{\mu^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \gamma K(\bar{\alpha}_s(\mu^2/\bar{\mu}^2, \alpha_s(\mu), \epsilon)) \right]\}
\]

• N=4 SYM: $\beta=0$, so $\alpha_s(\mu) = \bar{\alpha}_s = \text{constant}$, Running coupling in $d=4-2\epsilon$ has only trivial (engineering) dependence on scale $\mu$:

\[
\bar{\alpha}_s(\frac{\mu^2}{\bar{\mu}^2}, \epsilon) = \alpha_s \times \left( \frac{\mu^2}{\bar{\mu}^2} \right)^\epsilon (4\pi \epsilon \gamma)^\epsilon
\]

which makes it simple to perform integrals over $\xi$, $\bar{\mu}$
Sudakov form factor in planar N=4 SYM

- Expand $K$, $\gamma_K$, $G$ in terms of

$$a \equiv \frac{N_c \alpha_s}{2\pi} (4\pi e^{-\gamma})^\epsilon$$

$$K(\alpha_s, \epsilon) = \sum_{l=1}^{\infty} \frac{a^l}{2l\epsilon} \tilde{\gamma}_K^{(l)}$$

$$\gamma_K(\tilde{\alpha}_s(\mu^2/\bar{\mu}^2), \alpha_s(\mu), \epsilon)) = \sum_{l=1}^{\infty} a^l \left( \frac{\mu^2}{\bar{\mu}^2} \right)^{l\epsilon} \tilde{\gamma}_K^{(l)}$$

$$G(-1, \bar{\alpha}_s(\mu^2/\xi^2), \alpha_s(\mu), \epsilon), \epsilon) = \sum_{l=1}^{\infty} a^l \left( \frac{\mu^2}{\xi^2} \right)^{l\epsilon} \tilde{g}_0^{(l)}$$

- Perform integrals over $\xi$, $\bar{\mu}$

$$\mathcal{M}[g^g\rightarrow 1](Q^2/\mu^2, \alpha_s(\mu), \epsilon) = \exp\left[ -\frac{1}{4} \sum_{l=1}^{\infty} a^l \left( \frac{\mu^2}{Q^2} \right)^{l\epsilon} \left( \frac{\tilde{\gamma}_K^{(l)}}{(l\epsilon)}^2 + \frac{2\tilde{g}_0^{(l)}}{l\epsilon} \right) \right]$$
General amplitude in planar N=4 SYM

\[ M_n = \prod_{i=1}^{n} \left[ M_{gg-1} \left( \frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon \right) \right]^{1/2} \times h_n \left( k_i, \mu, \alpha_s, \epsilon \right) \]

\[ \Rightarrow M_n = 1 + \sum_{L=1}^{\infty} a^L M_n^{(L)} = \exp \left[ -\frac{1}{8} \sum_{l=1}^{\infty} a^l \left( \frac{\hat{\gamma}_{K}^{(l)}}{l \epsilon} + \frac{2 \hat{G}_0^{(l)}}{l \epsilon} \right) \sum_{i=1}^{n} \left( \frac{\mu^2}{-s_{i,i+1}} \right)^{l \epsilon} \right] \times h_n \]

which we can rewrite as

\[ M_n = \exp \left[ \sum_{l=1}^{\infty} a^l \left( f^{(l)}(\epsilon) M_n^{(1)}(l \epsilon) + h_n^{(l)}(\epsilon, s_{i,i+1}) \right) \right] \]

where \( f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)} \)

with

\[ f_0^{(l)} = \frac{1}{4} \hat{\gamma}_K^{(l)} \]
\[ f_1^{(l)} = \frac{l}{2} \hat{G}_0^{(l)} \]
\[ f_2^{(l)} = \text{???} \]

Insert result for form factor into this looks like the one-loop amplitude, but with \( \epsilon \) shifted to \( (l \epsilon) \), up to finite terms

"mixes" with \( h_n^{(l)} \)
Exponentiation in planar N=4 SYM

• **Miracle:** In planar N=4 SYM the finite terms also exponentiate. That is, the hard remainder function $h_n^{(l)}$ defined by

\[
\mathcal{M}_n = \exp\left[ \sum_{l=1}^{\infty} \alpha^l \left( f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + h_n^{(l)}(\epsilon, s_{i,i+1}) \right) \right]
\]

is also a series of simple constants, $C^{(l)}$ [for MHV amplitudes]:

\[
\mathcal{M}_n = \exp\left[ \sum_{l=1}^{\infty} \alpha^l \left( f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon) \right) \right]
\]

where $E_n^{(l)}(\epsilon) \to 0$ as $\epsilon \to 0$

Evidence based so far on two loops ($n=4,5$, plus collinear limits) and three loops (for $n=4$)

In contrast, for QCD, and non-planar N=4 SYM, two-loop amplitudes have been computed, and the hard remainders are a polylogarithmic mess!
Exponentiation at two loops

• The general formula,

$$M_n = 1 + \sum_{L=1}^{\infty} \alpha^L M_n^{(L)} = \exp \left[ \sum_{l=1}^{\infty} \alpha^l \left( f^{(l)}(\epsilon) M^{(1)}_n(l\epsilon) + C^{(l)} + E^{(l)}_n(\epsilon) \right) \right]$$

implies at two loops:

$$M_n^{(2)}(\epsilon) = \frac{1}{2} \left[ M_n^{(1)}(\epsilon) \right]^2 + f^{(2)}(\epsilon) M_n^{(1)}(2\epsilon) + C^{(2)} + E^{(2)}_n(\epsilon)$$

• To check at \( n=4 \), need to evaluate just 2 integrals:

$$\begin{align*}
\text{Smirnov, hep-ph/0111160} \\
\frac{1}{\epsilon^4}, \frac{1}{\epsilon^3}, \frac{1}{\epsilon^2}, \frac{1}{\epsilon}, \epsilon^0 \\
\text{elementary} \\
\frac{1}{\epsilon^2}, \frac{1}{\epsilon}, \epsilon^0, \epsilon, \epsilon^2
\end{align*}$$
Two-loop exponentiation & collinear limits

- Evidence for $n > 4$: Use limits as 2 momenta become collinear:
  
  \[ k_a \to z k_P \]
  
  \[ k_b \to (1 - z) k_P \]

- Tree amplitude behavior:

- One-loop behavior:

- Two-loop behavior:
Two-loop splitting amplitude iteration

• In N=4 SYM, all helicity configurations are equivalent, can write

\[
\text{Split}^{(l)}(\lambda_P, \lambda_a, \lambda_b) = r_S^{(l)}(z, s_{ab}, \epsilon) \times \text{Split}^{(0)}(\lambda_P, \lambda_a, \lambda_b)
\]

• Two-loop splitting amplitude obeys:

\[
r_S^{(2)}(\epsilon) = \frac{1}{2} \left[ r_S^{(1)}(\epsilon) \right]^2 + f^{(2)}(\epsilon)r_S^{(1)}(2\epsilon) + \mathcal{O}(\epsilon)
\]

Consistent with the \textit{n}-point amplitude ansatz

\[
M_n^{(2)}(\epsilon) = \frac{1}{2} \left[ M_n^{(1)}(\epsilon) \right]^2 + f^{(2)}(\epsilon)M_n^{(1)}(2\epsilon) + C^{(2)} + E_n^{(2)}(\epsilon)
\]

and fixes

\[
f_0^{(2)} = -\zeta_2 \quad f_1^{(2)} = -\zeta_3 \quad f_2^{(2)} = -\zeta_4 \quad C^{(2)} = \frac{-(\zeta_2)^2}{2}
\]

n-point information required to separate these two

Note: by definition \( f_0^{(1)} = 1, f_1^{(1)} = f_2^{(1)} = C^{(1)} = E_n^{(1)}(\epsilon) = 0 \)
Exponentiation at three loops

• L-loop formula implies at three loops

\[
M_n^{(3)}(\epsilon) = -\frac{1}{3}[M_n^{(1)}(\epsilon)]^3 + M_n^{(1)}(\epsilon)M_n^{(2)}(\epsilon) + f^{(3)}(\epsilon)M_n^{(1)}(3\epsilon) + C^{(3)} + E_n^{(3)}(\epsilon)
\]

• To check at \( n=4 \), need to evaluate just 4 integrals:

\[
\begin{align*}
\frac{1}{\epsilon^6}, \frac{1}{\epsilon^5}, \frac{1}{\epsilon^4}, \frac{1}{\epsilon^3}, \frac{1}{\epsilon^2}, \frac{1}{\epsilon}, \epsilon^0, \\
\frac{1}{\epsilon^4}, \frac{1}{\epsilon^3}, \frac{1}{\epsilon^2}, \frac{1}{\epsilon}, \epsilon^0, \epsilon, \epsilon^2, \\
\frac{1}{\epsilon^2}, \frac{1}{\epsilon}, \epsilon^0, \epsilon, \epsilon^2, \epsilon^3, \epsilon^4
\end{align*}
\]

Smirnov, hep-ph/0305142

Use Mellin-Barnes integration method
Harmonic polylogarithms

- Integrals are all transcendental functions of \( x = \frac{-s}{t} \)

- Expressed in terms of harmonic polylogarithms (HPLs)
  \( H_{a_1 a_2 \ldots a_n}(x) \equiv H(a_1, a_2, \ldots, a_n; x) \)

  with indices \( a_i \in \{0, 1\} \),

  defined recursively by:

  \[
  H_{a_1 a_2 \ldots a_n}(x) = \int_0^x dt \, f_{a_1}(t) \, H_{a_2 \ldots a_n}(t)
  \]

  with

  \[
  f_1(t) = \frac{1}{1-t}, \quad f_0(t) = \frac{1}{t}
  \]

- For \( w = 0, 1, 2, 3, 4 \), these HPLs can all be reduced to ordinary polylogarithms,

  \( \text{Li}_w(z) \) with \( z = x, \frac{1}{1-x}, \) or \( \frac{-x}{1-x} \)

- But here we need \( w = 5, 6 \) too


number of indices = weight \( w \)
Exponentiation at three loops

- Inserting the values of the integrals (including those with $s \leftrightarrow t$) into

$$M_4^{(3)}(\epsilon) = -\frac{1}{3}M_4^{(1)}(\epsilon)^3 + M_4^{(1)}(\epsilon)M_4^{(2)}(\epsilon) + f^{(3)}(\epsilon)M_4^{(1)}(3\epsilon) + C^{(3)} + E_4^{(3)}(\epsilon)$$

and using HPL identities relating $1/x \leftrightarrow x$, etc., we verify the relation, and extract

$$f_0^{(3)} = \frac{11}{2}\zeta_4 \quad f_1^{(3)} = 6\zeta_5 + 5\zeta_2\zeta_3 \quad f_2^{(3)} = c_1\zeta_6 + c_2\zeta_3^2$$

$$C^{(3)} = \left(\frac{341}{216} + \frac{2}{9}c_1\right)\zeta_6 + \left(-\frac{17}{9} + \frac{2}{9}c_2\right)\zeta_2$$

n-point information still required to separate
“Leading transcendentality” relation between QCD and N=4 SYM

- **KLOV** (Kotikov, Lipatov, Onishchenko, Velizhanin, hep-th/0404092) noticed (at 2 loops) a remarkable relation (miracle) between kernels for:
  - BFKL evolution (strong rapidity ordering)
  - DGLAP evolution (pdf evolution = strong collinear ordering)

in QCD and N=4 SYM:
- Set fermionic color factor $C_F = C_A$ in the QCD result and keep only the “leading transcendentality” terms. They coincide with the full N=4 SYM result (even though theories differ by scalars)
- Conversely, N=4 SYM results predict pieces of the QCD result

- “transcendentality”:
  - $1$ for $\pi$
  - $n$ for $\zeta_n = \text{Li}_n(1)$

similar counting for HPLs and for related harmonic sums used to describe DGLAP kernels
3-loop DGLAP splitting functions $P(x)$ in QCD

Related by a Mellin transform to the anomalous dimensions $\gamma(j)$ of leading-twist operators with spin $j$

$$\gamma(j) \equiv -\int_0^1 dx x^{j-1} P(x)$$


- KLOV obtained the N=4 SYM results by keeping only the “leading transcendentality” terms of MVV

<table>
<thead>
<tr>
<th></th>
<th>tree</th>
<th>1-loop</th>
<th>2-loop</th>
<th>3-loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q\gamma$</td>
<td>1</td>
<td>3</td>
<td>25</td>
<td>359</td>
</tr>
<tr>
<td>$g\gamma$</td>
<td>2</td>
<td>17</td>
<td>345</td>
<td></td>
</tr>
<tr>
<td>$h\gamma$</td>
<td>2</td>
<td></td>
<td>56</td>
<td></td>
</tr>
<tr>
<td>$qW$</td>
<td>1</td>
<td>3</td>
<td>32</td>
<td>589</td>
</tr>
<tr>
<td>$q\phi$</td>
<td>1</td>
<td>23</td>
<td>696</td>
<td></td>
</tr>
<tr>
<td>$g\phi$</td>
<td>1</td>
<td>8</td>
<td>218</td>
<td>6378</td>
</tr>
<tr>
<td>$h\phi$</td>
<td>1</td>
<td>33</td>
<td>1184</td>
<td></td>
</tr>
<tr>
<td>sum</td>
<td>3</td>
<td>18</td>
<td>350</td>
<td>9607</td>
</tr>
</tbody>
</table>

Table 1. The number of diagrams employed in our calculation of the three-loop splitting functions.
3-loop planar N=4 amplitude checks QCD

• From the value of the $1/\varepsilon^2$ pole in the scattering amplitude, we can check the KLOV observation, plus the MVV computation, in the large-spin $j$ limit of the leading-twist anomalous dimensions $\gamma(j)$ ($x \to 1$ limit of the $x$-space DGLAP kernel), also known as the soft or cusp anomalous dimension:

$$P_{a_a, x \to 1}^{(n)}(x) = \frac{A_{n+1}^a}{(1-x)_+} + B_{n+1}^a \delta(1-x) + C_{n+1}^a \ln(1-x) + O(1)$$

or

$$\gamma(j) = \frac{1}{2} \gamma_K(\alpha_s) (\ln(j) + \gamma_e) - B(\alpha_s) + O(\ln(j)/j)$$

where

$$A(\alpha_s) = \frac{1}{2} \gamma_K(\alpha_s)$$

Korchemsky (1989); Korchemsky, Marchesini (1993)
3-loop planar N=4 check (cont.)

\[ A_1^q = 4 C_F \]
\[ A_2^q = 8 C_F \left( \frac{67}{18} - \zeta_2 \right) C_A - \frac{5}{9} n_f \]
\[ A_3^q = 16 C_F C_A^2 \left[ \frac{245}{24} - \frac{67}{9} \zeta_2 + \frac{11}{6} \zeta_3 + \frac{11}{5} \zeta_2^2 \right] + 16 C_F^2 n_f \left[ -\frac{55}{24} + 2 \zeta_3 \right] + 16 C_F C_A n_f \left[ -\frac{209}{108} + \frac{10}{9} \zeta_2 - \frac{7}{3} \zeta_3 \right] + 16 C_F n_f^2 \left[ -\frac{1}{27} \right]. \]

MVV \( x \to 1 \) limit:

\[ f_0^{(1)} = 1 \]
\[ f_0^{(2)} = -\zeta_2 \]
\[ f_0^{(3)} = \frac{11}{2} \zeta_4 = \frac{11}{5} (\zeta_2)^2 \]

only 4 integrals required
3-loop planar N=4 check (cont.)

- $1/\varepsilon$ pole is sensitive to $G$ in Sudakov form factor.
- We can apply KLOV’s prescription here as well, to predict the leading transcendentality terms in the three-loop QCD form factor:

$$ f_1^{(3)} = 6\zeta_5 + 5\zeta_2 \zeta_3 \Rightarrow G_0^{(3)} = C_A^3 \left( 4\zeta_5 + \frac{10}{3} \zeta_2 \zeta_3 \right) $$

- Prediction confirmed by MVV (hep-ph/0508055)

$$ G_3^g = C_A^3 \left( \frac{-373975}{729} - \frac{27320}{81} \zeta_2 + \frac{4096}{27} \zeta_3 + \frac{1276}{15} \zeta_2^2 + \frac{80}{3} \zeta_2 \zeta_3 + 32 \zeta_5 \right) $$

$$ + C_A^2 n_f \left( \frac{266072}{729} + \frac{7328}{81} \zeta_2 + \frac{56}{9} \zeta_3 - \frac{328}{15} \zeta_2^2 \right) + C_A C_F n_f \left( \frac{3833}{27} + 8 \zeta_2 \right) $$

$$ - \frac{752}{9} \zeta_3 + \frac{32}{5} \zeta_2 \right) - 4C_F^2 n_f + C_A n_f^2 \left( - \frac{28114}{729} - \frac{160}{27} \zeta_2 - \frac{256}{27} \zeta_3 \right) $$

$$ + C_F n_f^2 \left( - \frac{104}{3} + \frac{64}{3} \zeta_3 \right) . $$
Integrability & anomalous dimensions

- The dilatation operator for N=4 SYM, treated as a Hamiltonian, is integrable at one loop.
- E.g. in $SU(2)$ subsector, $\text{tr}( Z^L X^j )$, it is a Heisenberg model

Minahan, Zarembo, hep-th/0212208; Beisert, Staudacher, hep-th/0307042

- Much accumulating evidence of multi-loop integrability in various sectors of the theory

Beisert, Dippel, Eden, Jarczak, Kristjansen, Rej, Serban, Staudacher, Zwiebel, Belitsky, Gorsky, Korchemsky, …

including $sl(2)$ sector, $\text{tr}( D^j Z^L )$, directly at two loops

Eden, Staudacher, hep-th/0603157

A conjectured all-orders asymptotic (large $j$) Bethe ansatz has been obtained by deforming the “spectral parameter” $u$ to $x$:

$$ u \pm \frac{i}{2} = x^{\pm} + \frac{g^2}{2x^{\pm}} $$

All-orders proposal

The all-orders asymptotic Bethe ansatz leads to the following proposal for the soft/cusp anomalous dimension in $N=4$ SYM:

$$f(g) = 4g^2 - 16g^4 \int_0^\infty dt \hat{\sigma}(t) \frac{J_1(\sqrt{2} gt)}{\sqrt{2} gt}$$

where

$$\hat{\sigma}(t) = \frac{t}{e^t - 1} \left[ \frac{J_1(\sqrt{2} g t)}{\sqrt{2} g t} - 2g^2 \int_0^\infty dt' \hat{K}(\sqrt{2} g t, \sqrt{2} g t') \hat{\sigma}(t') \right]$$

is the solution to an integral equation with Bessel-function kernel

$$\hat{K}(t, t') = \frac{J_1(t) J_0(t') - J_0(t) J_1(t')}{t - t'}$$

Perturbative expansion:

$$f(g) = 4g^2 - \frac{2}{3} \pi^2 g^4 + \frac{11}{45} \pi^4 g^6 - \left( \frac{73}{630} \pi^6 - 4 \zeta(3)^2 \right) g^8 + \ldots$$
Beyond three loops

Bethe ansatz “wrapping problem” when interaction range exceeds spin chain length, implies proposal needs checking via other methods, e.g. gluon scattering amplitudes – particularly at 4 loops.

Recently two programs have been written to automate the extraction of $1/\varepsilon$ poles from Mellin-Barnes integrals, and set up numerical integration over the multiple inversion contours.

Anastasiou, Daleo, hep-ph/0511176; this workshop;
Czakon, hep-ph/0511200

Numerics should be enough to check four-loop ansatze.

Z. Bern et al, in progress
Numerical two-loop check for $n=5$

Collinear limits are highly suggestive, but not quite a proof.

Using unitarity, first in $D=4$, later in $D=4-2\epsilon$, the two-loop $n=5$ amplitude was found to be:

$$s_{12}^2 s_{23} + s_{12}^2 s_{51} + s_{12} s_{34} s_{45} (q - k_i)^2$$

$$+ R \left[ - \frac{s_{12}}{s_{34} s_{45}} \frac{d_{++}}{s_{51}^2} + \frac{d_{++}}{s_{23}} \right]$$

$$+ \frac{d_{+..}}{s_{23} s_{51}} (q - k_i)^2 + 2$$

+ cyclic

$$R = \varepsilon(k_1, k_2, k_3, k_4)$$

$$\times s_{12} s_{23} s_{34} s_{45} s_{51} / \det(s_{ij})|_{i,j=1,2,3,4}$$

Even and odd terms checked numerically with aid of

Bern, Czakon, Kosower, Roiban, Smirnov, hep-th/0604074

Cachazo, Spradlin, Volovich, hep-th/0602228

Bern, Rozowsky, Yan, hep-ph/9706392

Bern, Czakon, hep-ph/0511200
Conclusions & Outlook

- N=4 SYM captures most singular infrared behavior of QCD.
- Finite terms exponentiate in a very similar way to the IR divergent ones, in the planar, large $N_c$ limit.
- How is this related to the AdS/CFT correspondence?
- "Leading transcendentality" relations for some quantities. Why?
- Is the Eden-Staudacher all-order prediction for the soft anomalous dimension correct at 4 loops?
Extra Slides
Simpler ways to check ansatz?

One can apply suitable differential operators to terms in the ansatz, which reduce their degree of infrared divergence

Cachazo, Spradlin, Volovich, hep-th/0601031

At two loops, \( \mathcal{L}^{(2)} = \left[ \frac{d^2}{d(\ln x)^2} - \epsilon^2 \right]^3 \) annihilates

\[
(st)^\epsilon M_4^{(2)}(\epsilon) = \frac{2}{\epsilon^4} - \frac{1}{\epsilon^2} \left[ \frac{1}{2} \ln^2(-x) + \frac{5}{4} \pi^2 \right] + \mathcal{O}(\epsilon^{-1}),
\]

\[
(st)^\epsilon \left(\frac{1}{2} M_4^{(1)}(\epsilon) \right)^2 = \frac{2}{\epsilon^4} - \frac{1}{\epsilon^2} \left[ \frac{1}{2} \ln^2(-x) + \frac{4}{3} \pi^2 \right] + \mathcal{O}(\epsilon^{-1}).
\]

and greatly simplifies the MB integral evaluation.

Verifies two-loop ansatz up to functions in the kernel of \( \mathcal{L}^{(2)} \)

\[
\frac{1}{(st)^\epsilon} K(x, \epsilon) = C + E \ln^2(-x) + F \ln^4(-x) + \mathcal{O}(\epsilon),
\]

Generalization to multi-loops may also be quite useful.
Other theories

Two classes of (large $N_c$) conformal gauge theories “inherit” the same large $N_c$ perturbative amplitude properties from N=4 SYM:

1. Theories obtained by orbifold projection
   – product groups, matter in particular bi-fundamental rep’s

   Bershadsky, Johansen, hep-th/9803249

2. The N=1 supersymmetric “beta-deformed” conformal theory
   – same field content as N=4 SYM, but superpotential is modified:

   \[ ig \text{ Tr}(\Phi_1 \Phi_2 \Phi_3 - \Phi_1 \Phi_3 \Phi_2) \rightarrow ig \text{ Tr}(e^{i\pi/2} \Phi_1 \Phi_2 \Phi_3 - e^{-i\pi/2} \Phi_1 \Phi_3 \Phi_2) \]

   Supergravity dual known for this case, deformation of AdS$_5 \times$ S$^5$

   Lunin, Maldacena, hep-th/0502086

Breakdown of inheritance at five loops (!?) for more general marginal perturbations of N=4 SYM?

Khoze, hep-th/0512194
How are QCD and N=4 SYM related?

At tree-level they are essentially identical

Consider a tree amplitude for $n$ gluons. Fermions and scalars cannot appear because they are produced in pairs.

Hence the amplitude is the same in QCD and N=4 SYM. The QCD tree amplitude “secretly” obeys all identities of N=4 supersymmetry:
At loop-level, QCD and N=4 SYM differ

However, it is profitable to rearrange the QCD computation to exploit supersymmetry.

- Gluon loop

\[ \text{contains most of the } \frac{1}{\epsilon} \text{ poles} \]

Perhaps best virtue of decomposition: gives excuse to compute beautiful, simple amplitudes in N=4 SYM -- consider them as one of 3 components of practical QCD amplitudes.

- N=4 SYM
- N=1 multiplet
- Scalar loop --- no SUSY, but also no spin tangles
The “tennis court” integral

\( (l + k_4)^2 = I_{4}^{(3b)}(s, t) = -\frac{1}{(-s)^{1+3\varepsilon} t^2} \sum_{j=0}^{6} \frac{c_j(x, L)}{\varepsilon^j} \)

\( x = -t/s, \ L = \ln(s/t) \)

\[
c_6 = \frac{16}{9}, \ c_5 = \frac{13}{6} L, \ c_4 = \frac{1}{2} L^2 - \frac{19}{12} \pi^2, \\
c_3 = \frac{5}{2} \left[ H_{0,0,1}(x) + L H_{0,1}(x) \right] + \frac{5}{4} \left[ L^2 + \pi^2 \right] H_1(x) \\
-\frac{7}{12} L^3 - \frac{157}{72} L \pi^2 - \frac{241}{18} \zeta_3, \\
c_2 = \frac{1}{2} \left[ 11 H_{0,0,0,1}(x) - 5 H_{0,0,1,1}(x) - 5 H_{0,1,0,1}(x) - 5 H_{1,0,0,1}(x) \right] \\
+\frac{1}{2} L \left[ 14 H_{0,0,1}(x) - 5 H_{0,1,1}(x) - 5 H_{1,0,1}(x) \right] + \frac{1}{4} L^2 \left[ 17 H_{0,1}(x) - 5 H_{1,1}(x) \right] \\
+\frac{4}{3} \pi^2 H_{0,1}(x) - \frac{5}{4} \pi^2 H_{1,1}(x) + \frac{5}{3} L^3 H_1(x) + \frac{25}{12} L \pi^2 H_1(x) \\
-\frac{41}{3} L \zeta_3 + \frac{5}{2} H_1(x) \zeta_3 - \frac{1}{3} L^4 - \frac{1}{4} L^2 \pi^2 + \frac{2429}{6480} \pi^4 ,
\]
“Tennis court” integral (cont.)

\[ c_1 = \frac{1}{2} \left[ -55 H_{0,0,0,0,1}(x) - 59 H_{0,0,0,1,1}(x) - 31 H_{0,0,1,0,1}(x) + 5 H_{0,0,1,1,1}(x) 
- 3 H_{0,1,0,0,1}(x) + 5 H_{0,1,0,1,1}(x) + 3 H_{0,1,1,0,1}(x) + 25 H_{1,0,0,0,1}(x) 
+ \frac{1}{2} L \left[ 22 H_{0,0,0,1}(x) - 46 H_{0,0,1,1}(x) - 18 H_{0,1,0,1}(x) + 5 H_{0,1,1,1}(x) 
+ 10 H_{1,0,0,1}(x) + 5 H_{1,0,1,1}(x) + 5 H_{1,1,0,1}(x) \right] 
+ \frac{1}{4} L^2 \left[ 64 H_{0,0,1}(x) - 33 H_{0,1,1}(x) - 5 H_{1,0,1}(x) + 5 H_{1,1,1}(x) \right] 
+ \frac{1}{24} \pi^2 \left[ 25 H_{0,0,1}(x) - 128 H_{0,1,1}(x) + 40 H_{1,0,1}(x) + 30 H_{1,1,1}(x) \right] 
+ \frac{1}{12} L^3 \left[ 71 H_{0,1}(x) - 20 H_{1,1}(x) \right] 
+ \frac{1}{24} L^2 \left[ 153 H_{0,1}(x) - 50 H_{1,1}(x) \right] + \frac{1}{2} \left[ 8 H_{0,1}(x) - 5 H_{1,1}(x) \right] \zeta_3 
+ \frac{43}{48} L^4 H_1(x) + \frac{71}{48} L^2 \pi^2 H_1(x) - \frac{5}{144} \pi^4 H_1(x) - \frac{5}{2} L H_1(x) \zeta_3 + \frac{7}{48} L^5 
+ \frac{227}{144} L^3 \pi^2 + \frac{13}{4} L^2 \zeta_3 + \frac{10913}{8640} L \pi^4 + \frac{3257}{216} \pi^2 \zeta_3 - \frac{889}{10} \zeta_5, \]
“Tennis court” integral (cont.)

\[ c_0 = \frac{1}{2} \left[ 379H_{0,0,0,0,0,1}(x) + 343H_{0,0,0,0,1,1}(x) + 419H_{0,0,0,1,0,1}(x) + 347H_{0,0,1,1,1,1}(x) + 355H_{0,1,0,0,0,1}(x) + 175H_{0,1,0,0,1,1}(x) + 223H_{0,1,0,1,0,1}(x) - 5H_{0,1,0,1,1,1}(x) + 151H_{1,0,0,0,0,1}(x) + 3H_{1,0,0,0,1,1}(x) + 51H_{1,0,0,1,0,1}(x) - 5H_{1,0,0,1,1,1}(x) + 99H_{1,0,1,0,0,1}(x) - 5H_{1,0,1,0,1,1}(x) - 5H_{1,0,1,1,0,1}(x) - 193H_{1,0,1,0,1,1}(x) - 169H_{1,1,0,0,0,1}(x) - 121H_{1,1,0,0,1,1}(x) - 73H_{1,0,1,0,0,1}(x) - 5H_{1,1,0,0,1,1}(x) - 5H_{1,1,0,0,1,1}(x) - 25H_{1,1,0,0,0,1}(x) - 5H_{1,1,0,0,1,1}(x) - 5H_{1,1,0,1,0,1}(x) - 5H_{1,1,0,1,1,1}(x) + \frac{1}{2}L \left[ 98H_{0,0,0,0,1}(x) - 22H_{0,0,0,1,1}(x) + 98H_{0,1,0,0,1}(x) + 238H_{0,1,0,1,1}(x) + 78H_{0,1,0,0,1}(x) + 66H_{0,1,1,0,1}(x) + 114H_{0,1,1,1,1}(x) - 5H_{0,1,1,1,1}(x) - 82H_{1,0,0,0,1}(x) - 106H_{1,0,0,1,1}(x) - 58H_{1,0,1,0,1}(x) - 5H_{1,0,1,1,1}(x) - 10H_{1,1,0,0,1}(x) - 5H_{1,1,0,1,1}(x) - 5H_{1,1,1,0,1}(x) + \frac{1}{4}L^3 \left[ 124H_{0,0,0,1}(x) - 208H_{0,0,1,1}(x) - 44H_{0,1,0,1}(x) + 129H_{0,1,1,1}(x) - 20H_{1,0,0,1}(x) - 43H_{1,0,1,1}(x) + 5H_{1,1,0,1}(x) - 5H_{1,1,1,1}(x) \right] + \frac{1}{24} \pi^2 \left[ 183H_{0,0,0,1}(x) - 121H_{0,0,1,1}(x) + 375H_{0,1,0,1}(x) + 704H_{0,1,1,1}(x) + 31H_{0,0,0,1}(x) - 328H_{0,1,0,1}(x) - 40H_{1,0,1,1}(x) - 30H_{1,1,1,1}(x) \right] + \frac{1}{12}L^3 \left[ 260H_{0,0,1}(x) - 215H_{0,1,1}(x) - 7H_{0,1,0,1}(x) + 20H_{1,1,0,1}(x) \right] + \frac{1}{24} \pi^2 \left[ 326H_{0,0,0,1}(x) - 633H_{0,0,1,1}(x) + 177H_{0,1,0,1}(x) + 50H_{1,0,1,1}(x) \right] - \frac{1}{2} \pi^2 \left[ -3LH_{0,1}(x) - 5LH_{1,1}(x) + 165H_{0,0,0,1}(x) + 104H_{0,0,1,1}(x) - 68H_{0,1,0,1}(x) + 5H_{1,0,1}(x) \right] + \frac{1}{48}L^3 \left[ 309H_{0,0,1}(x) - 43H_{0,1,1}(x) + \frac{1}{48} \pi^2 \left[ 725H_{0,1}(x) - 71H_{1,1}(x) \right] + \frac{1}{720} \pi^2 \left[ 1848H_{0,1}(x) + 25H_{1,1}(x) \right] \right]\]