Parton Shower as QCD Prediction

Shower Evolution, Matching at LO and NLO level

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LoopFest V Radiative Corrections for the ILC: Multi-loops and Multi-legs

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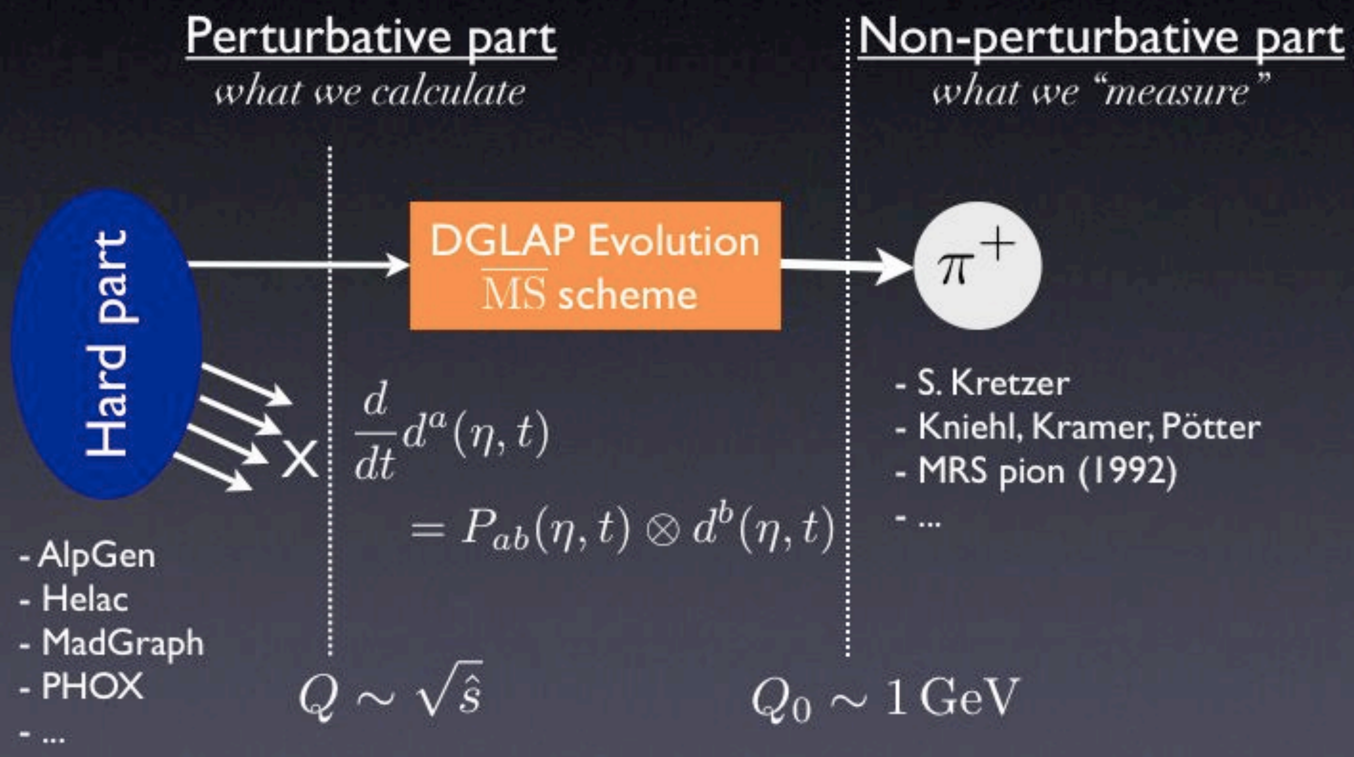
Introduction

Structure of the one-hadron inclusive cross section

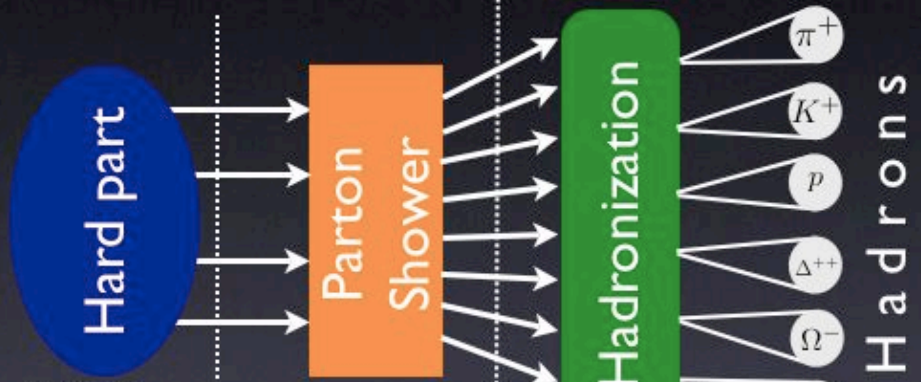


Introduction

Structure of the Monte Carlo algorithm

Perturbative part
what we calculate

Non-perturbative part
what we "measure"



- AlpGen
- Helac
- MadGraph
- MFCM
- NLOJET++
- ...

- Herwig(++)
- Pythia(++)
- Ariadne
- ...

- Lund model
- Cascade model
- ...

$Q \sim \sqrt{\hat{s}}$ $\quad Q_0 \sim 1\,\mathrm{GeV}$

Parton shower algorithms:

- QCD inspired model
- Considers higher order (LL)
- Not predictive: The scales and unphysical parameters are rather uncontrolled.
- Poor soft gluon treatment
- Matching to exact matrix elements is a "real challenge".
- Important tool for detector simulation.
- Crude approximation but with tuning the performance is very good.

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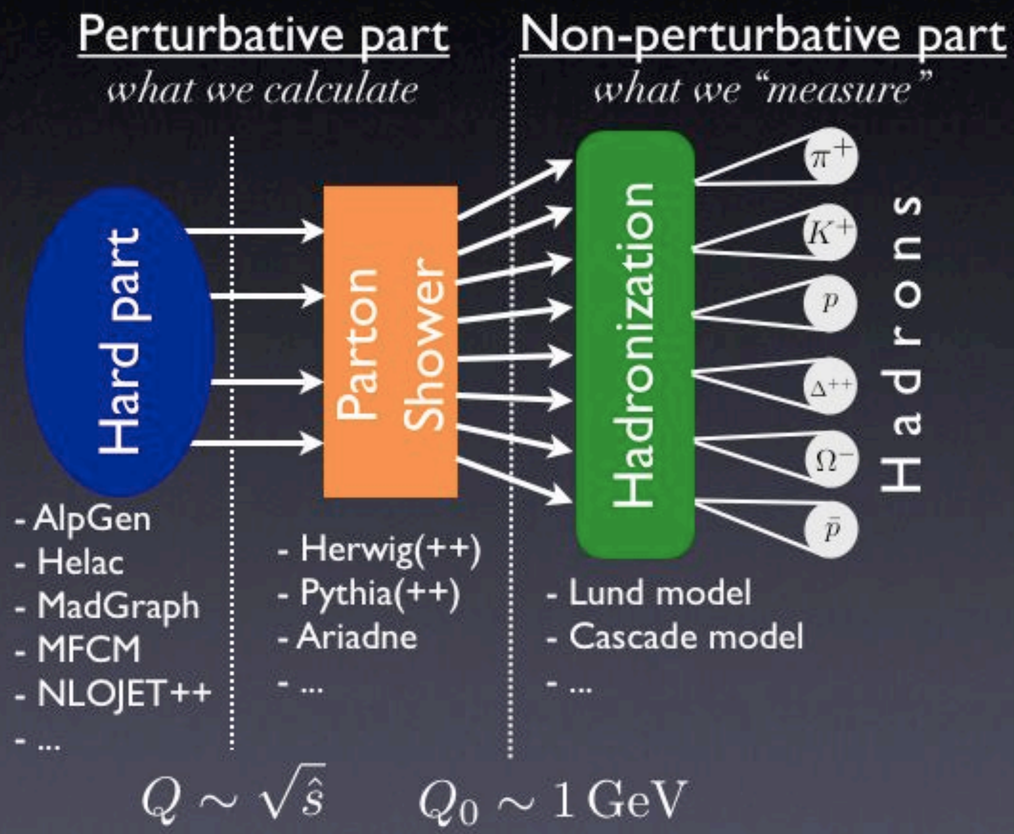
Structure of the Monte Carlo algorithm



A precison parton shower knows:

- Maximal phase space coverage
- Lorentz invariant/covariant
- Better soft gluon treatment (NLL); possibly exact treatment.
- Color evolution.
- No ambiguous parameters; only the infrared cutoff parameter.
- Tuning is allowed only in the hadronization part.
- Matching to exact matrix element should be straightforward.
- Adding higher order contributions should be "straightforward".

Configuration Space

An m-parton configuration is

$$\{p,f,c,d\}_m \equiv \{\eta_{\rm a} p_A, a, c_A, d_A, \eta_{\rm b} p_B, b, c_B, d_B, ..., p_m, f_m, c_m, d_m\}$$

Basis vector in the configuration space: $\big|\{p,f,c,d\}_m\big)$

Normalization:

$$\begin{aligned}(\{p',f',c',d'\}_{m'}|\{p,f,c,d\}_m) = {} & \delta_{mm'}\,\delta_{a,a'}\,\delta_{c_A c'_A}\,\delta_{d_A d'_A}\,\delta(\eta_{\rm a}-\eta'_{\rm a}) \\ & \times \delta_{b,b'}\,\delta_{c_B c'_B}\,\delta_{d_B d'_B}\,\delta(\eta_{\rm b}-\eta'_{\rm b}) \prod_{i=1}^{m} \delta_{f,f'_i}\,\delta_{c_i c'_i}\,\delta_{d_i d'_i}\,\delta^{(4)}(p_i-p'_i)\end{aligned}$$

Completeness relation:

$$1 = \sum_m \int [d\{p,f,c,d\}_m]\,|\{p,f,c,d\}_m)(\{p,f,c,d\}_m|$$

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Normalization:

$$\int [d\{p,f,c,d\}_m] \equiv \sum_a \int_0^1 d\eta_{\rm a} \sum_b \int_0^1 d\eta_{\rm b} \prod_{i=1}^m \left\{ \sum_{f_i} \int d^4 p_i \right\} \sum_{\{c,d\}_m}$$

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Configuration Space

A general state (e.g. jet function) is

$$(F| = \sum_m \int [d\{p,f,c,d\}_m]\, F(\{p,f,c,d\}_m) (\{p,f,c,d\}_m| \ .$$

The unit vector is

$$(1| = \sum_m \int [d\{p,f,c,d\}_m]\, (\{p,f,c,d\}_m| \ .$$

Completeness relation:

$$1 = \sum_m \int [d\{p,f,c,d\}_m]\, |\{p,f,c,d\}_m)(\{p,f,c,d\}_m|$$

Phase Space Integral

To define the phase space integral we have an operator

$$\Gamma = \sum_m \int \left[d\{p,f,c,d\}_m\right] \big|\{p,f,c,d\}_m\big)\big(\{p,f,c,d\}_m\big|$$
$$\times f_{a/A}(\eta_{\rm a},\mu_F^2)\, f_{b/B}(\eta_{\rm b},\mu_F^2)\, \frac{1}{2\eta_{\rm a}\eta_{\rm b} p_A\!\cdot\! p_B}\, \frac{1}{m!}$$
$$\times \prod_{i=1}^{m} \left\{ \frac{1}{(2\pi)^3} \delta_+(p_i^2) \right\} (2\pi)^4 \delta\Big(\eta_{\rm a} p_A + \eta_{\rm b} p_B - K - \sum_{i=1}^{m} p_i \Big)$$

Cross section in the configuration space

$$\big|\sigma_m\big) = \Gamma\big|\mathcal{M}_m\big) \quad \Rightarrow \quad \sigma_m[F_{m-\rm jet}] = \big(F_{m-\rm jet}\big|\Gamma\big|\mathcal{M}_m\big)$$

Color Flow at Quantum Level

The QCD matrix elements can be expanded on a color basis

$$\mathcal{M}(\{p,f\}_m)^{A_a,\ldots,A_m} = \sum_{\{c\}_m} V(\{c\}_m)^{A_a,\ldots,A_m} \mathcal{M}(\{p,f,c\}_m)$$

The color basis is based on open and closed color strings:

$$V(\{c\}_m)^{A_1,\ldots,A_m} = V(S_1)^{\{A\}_{[1]}} V(S_2)^{\{A\}_{[2]}} \ldots V(S_n)^{\{A\}_{[n]}}$$

The open sting is given by

$$V(S)^{\{A\}} = (N_c C_F^{n-2})^{-1/2} \left[t^{A_2} t^{A_3} \cdots t^{A_{n-1}}\right]_{A_1 A_n}$$

while the closed string is represented by

$$V(S)^{\{A\}} = \left(C_F^n \left[1 - \left(\frac{-1}{N_c^2 - 1}\right)^{n-1}\right]\right)^{-1/2} \mathrm{Tr}\left[t^{A_1} t^{A_2} \cdots t^{A_n}\right]$$

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The ope

$V(S$ ... $A_1 A_n$

while th

$V(S)^{\{A}$ ... $\ldots t^{A_n}]$

This basis is orthogonal up to the subleading color terms:

$$\left\langle \{c\}_m \middle| \{c\}_m \right\rangle = 1$$

$$\left\langle \{c\}_m \middle| \{c'\}_m \right\rangle = \mathcal{O}(1/N_{\mathrm{c}}^2)$$

Color flow at Quantum Level

Representing the amplitude in the configuration space

$$|\mathcal{M}_m) = \int \left[d\{p,f,c,d\}_m\right] |\{p,f,c,d\}_m)(\{p,f,c,d\}_m|\mathcal{M}_m)$$

where the coefficients are

$$(\{p,f,c,d\}_m|\mathcal{M}_m) = {}_{\rm c}\langle\{c\}_m|\{d\}_m\rangle_{\rm c}\ {}_{\rm s}\langle\mathcal{M}(\{p,f,c\}_m)|\mathcal{M}(\{p,f,d\}_m)\rangle_{\rm s}$$

Summing over all the possible color flow configurations

$$\sum_{\{c\}_m}\sum_{\{d\}_m}(\{p,f,c,d\}_m|\mathcal{M}_m) = {}_{\rm c,s}\langle\mathcal{M}(\{p,f\}_m)|\mathcal{M}(\{p,f\}_m)\rangle_{\rm c,s}$$

Parton Shower Evolution

We use an evolution variable e.g.: $\log \dfrac{Q^2}{\hat{p}_1 \cdot \hat{p}_2} = t \ \in \ [0, \infty]$

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$$U(t_3, t_1) = \underbrace{N(t_3, t_1)}_{\text{No-splitting part}} + \overbrace{\int_{t_1}^{t_3} dt_2\, U(t_3, t_2)\, \mathcal{H}(t_2)\, N(t_2, t_1)}^{\text{Splitting part}}$$

Preserves the normalization

$$(1|A(t_0)) = 1 \quad \Rightarrow \quad (1|U(t, t_0)|A(t_0)) = 1$$

No-splitting Operator

The operator $N(t', t)$ leaves the basis states $|\{p, f, c, d\}_m)$ unchanged

$$N(t', t)|\{p, f, c, d\}_m) = \underbrace{\Delta(\{p, f, c, d\}_m; t', t)}_{\text{Sudakov factor}}|\{p, f, c, d\}_m)$$

From the normalization $(1|U(t, t')|\{p, f, c, d\}_m) = 1$

$$1 = \Delta(\{p, f, c, d\}_m; t_3, t_1) + \int_{t_1}^{t_3} dt_2\ (1|\mathcal{H}(t_2)|\{p, f, c, d\}_m)\,\Delta(\{p, f, c, d\}_m; t_2, t_1)$$

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$$\Delta(\{p,f,c,d\}_m;t_2,t_1) = \exp\left(-\int_{t_1}^{t_2} dt\,\left(1\middle|\mathcal{H}(t)\middle|\{p,f,c,d\}_m\right)\right)$$

Splitting Operator

The splitting operator describes all the possible transitions that $|\{p, f, c, d\}_m) \rightarrow |\{\hat{p}, \hat{f}, \hat{c}, \hat{d}\}_{m+n})$

$$\mathcal{H}(t) = \underbrace{\mathcal{H}^{(0)}(t)}_{1\rightarrow 2} + \frac{\alpha_s}{2\pi} \underbrace{\mathcal{H}^{(1)}(t)}_{\substack{1\rightarrow 3 \\ 1\rightarrow 2 \text{ at 1-loop} \\ 2\rightarrow 4 \\ \cdots}} + \cdots$$

Since we are interested only at LL and NLL level we have only $1 \rightarrow 2$ splittings

Splitting Operator

$$
\begin{aligned}
&\left(\{\hat{p},\hat{f},\hat{c},\hat{d}\}_{m+1}\middle|\mathcal{H}(t)\middle|\{p,f,c,d\}_m\right)\\
&\qquad = \sum_{\substack{l,k\\k\neq l}} \int_0^1 \frac{dy}{y}\int_0^1 dz \int_0^{2\pi}\frac{d\phi}{2\pi}\,\frac{\alpha_s(Q^2e^{-t})}{2\pi}\,\delta\big(t+\log(T_{l,k}(p_l,p_k,z,y)/Q^2)\big)\\
&\qquad\quad \times \frac{\big\langle\{\hat{c}\}_{m+1}\big|\{\hat{d}\}_{m+1}\big\rangle}{\big\langle\{c\}_m\big|\{d\}_m\big\rangle}\left(\{\hat{c},\hat{d}\}_{m+1}\middle|\mathcal{S}_{l,k}(z,y,\hat{f}_{l_1},\hat{f}_{l_2})\middle|\{c,d\}_m\right)\\
&\qquad\quad \times \frac{\hat{\eta}_{\mathrm{a}}}{\eta_{\mathrm{a}}}\frac{f_{\hat{a}/A}(\hat{\eta}_{\mathrm{a}},\mu_F^2)}{f_{a/A}(\eta_{\mathrm{a}},\mu_F^2)}\,\frac{\hat{\eta}_{\mathrm{b}}}{\eta_{\mathrm{b}}}\frac{f_{\hat{b}/B}(\hat{\eta}_{\mathrm{b}},\mu_F^2)}{f_{b/B}(\eta_{\mathrm{b}},\mu_F^2)}\left(\{\hat{p},\hat{f}\}_{m+1}\middle|\mathcal{R}_{l,k}(z,y,\kappa_\perp)\middle|\{p,f\}_m\right)
\end{aligned}
$$

Splitting Operator

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\begin{aligned}
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&\qquad \times \frac{\big\langle\{\hat{c}\}_{m+1}\big|\{\hat{d}\}_{m+1}\big\rangle}{\big\langle\{c\}_m\big|\{d\}_m\big\rangle}\big(\{\hat{c},\hat{d}\}_{m+1}\big|\mathcal{S}_{l,k}(z,y,\hat{f}_{l_1},\hat{f}_{l_2})\big|\{c,d\}_m\big) \\
&\qquad \times \frac{\hat{\eta}_{\mathrm{a}}}{\eta_{\mathrm{a}}}\frac{f_{\hat{a}/A}(\hat{\eta}_{\mathrm{a}},\mu_F^2)}{f_{a/A}(\eta_{\mathrm{a}},\mu_F^2)}\,\frac{\hat{\eta}_{\mathrm{b}}}{\eta_{\mathrm{b}}}\frac{f_{\hat{b}/B}(\hat{\eta}_{\mathrm{b}},\mu_F^2)}{f_{b/B}(\eta_{\mathrm{b}},\mu_F^2)}\big(\{\hat{p},\hat{f}\}_{m+1}\big|\mathcal{R}_{l,k}(z,y,\kappa_\perp)\big|\{p,f\}_m\big)
\end{aligned}
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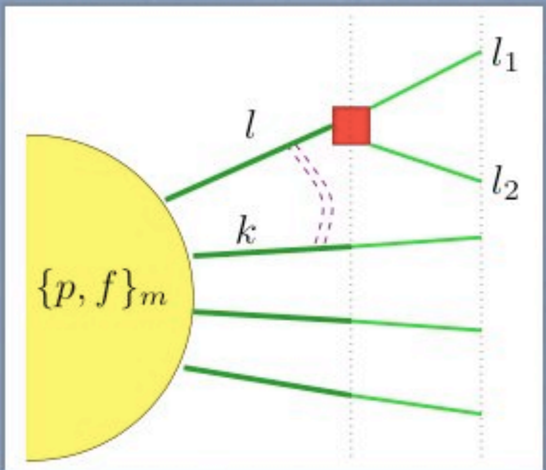

$$
\begin{aligned}
&\int\big[d\{\hat{c},\hat{d}\}_{m+1}\big]\big(\{\hat{c},\hat{d}\}_{m+1}\big|\mathcal{S}_{l,k}(z,y,\hat{f}_{l_1},\hat{f}_{l_2})\big|\{c,d\}_m\big) \\
&\qquad\times\frac{\big\langle\{\hat{c}\}_{m+1}\big|\{\hat{d}\}_{m+1}\big\rangle}{\big\langle\{c\}_m\big|\{d\}_m\big\rangle} \\
&\quad=\boxed{\frac{\big\langle\{c\}_m\big|\boldsymbol{T}_l\cdot\boldsymbol{T}_k\big|\{d\}_m\big\rangle}{-\boldsymbol{T}_l^2\,\big\langle\{c\}_m\big|\{d\}_m\big\rangle}}S_{l,k}(z,y,\hat{f}_{l_1},\hat{f}_{l_2})
\end{aligned}
$$

Splitting Operator

$$
\begin{aligned}
&\left(\{\hat{p},\hat{f},\hat{c},\hat{d}\}_{m+1}\middle|\mathcal{H}(t)\middle|\{p,f,c,d\}_m\right) \\
&\quad = \sum_{\substack{l,k \\ k\neq l}} \int_0^1 \frac{dy}{y}\int_0^1 dz \int_0^{2\pi}\frac{d\phi}{2\pi}\,\frac{\alpha_s\!\left(Q^2e^{-t}\right)}{2\pi}\,\delta\!\left(t+\log(T_{l,k}(p_l,p_k,z,y)/Q^2)\right) \\
&\quad \times \frac{\left\langle\{\hat{c}\}_{m+1}\middle|\{\hat{d}\}_{m+1}\right\rangle}{\left\langle\{c\}_m\middle|\{d\}_m\right\rangle}\left(\{\hat{c},\hat{d}\}_{m+1}\middle|\mathcal{S}_{l,k}(z,y,\hat{f}_{l_1},\hat{f}_{l_2})\middle|\{c,d\}_m\right) \\
&\quad \times \frac{\hat{\eta}_{\mathrm{a}}}{\eta_{\mathrm{a}}}\frac{f_{\hat{a}/A}(\hat{\eta}_{\mathrm{a}},\mu_F^2)}{f_{a/A}(\eta_{\mathrm{a}},\mu_F^2)}\,\frac{\hat{\eta}_{\mathrm{b}}}{\eta_{\mathrm{b}}}\frac{f_{\hat{b}/B}(\hat{\eta}_{\mathrm{b}},\mu_F^2)}{f_{b/B}(\eta_{\mathrm{b}},\mu_F^2)}\left(\{\hat{p},\hat{f}\}_{m+1}\middle|\mathcal{R}_{l,k}(z,y,\kappa_\perp)\middle|\{p,f\}_m\right)
\end{aligned}
$$

Sudakov parametrization of the new momenta:

$$\hat{p}_{l,1} = z p_l + y(1-z)p_k + k_\perp \qquad p_l + p_k = \hat{p}_{l,1} + \hat{p}_{l,2} + \hat{p}_k$$

$$\hat{p}_{l,2} = (1-z)p_l + yz p_k - k_\perp \qquad \hat{p}_{l,1}^2 = \hat{p}_{l,2}^2 = 0$$

$$\hat{p}_k = (1-y)p_k \qquad -k_\perp^2 = 2p_l\cdot p_k\, yz(1-z)$$

Splitting Operator

$$\left(\{\hat{p},\hat{f},\hat{c},\hat{d}\}_{m+1}\middle|\mathcal{H}(t)\middle|\{p,f,c,d\}_m\right)$$

$$= \sum_{\substack{l,k\\k\neq l}} \int_0^1 \frac{dy}{y}\int_0^1 dz \int_0^{2\pi}\frac{d\phi}{2\pi}\,\frac{\alpha_s(Q^2 e^{-t})}{2\pi}\,\delta\!\left(t+\log(T_{l,k}(p_l,p_k,z,y)/Q^2)\right)$$

$$\times\,\frac{\left\langle\{\hat{c}\}_{m+1}\middle|\{\hat{d}\}_{m+1}\right\rangle}{\left\langle\{c\}_m\middle|\{d\}_m\right\rangle}\left(\{\hat{c},\hat{d}\}_{m+1}\middle|\mathcal{S}_{l,k}(z,y,\hat{f}_{l_1},\hat{f}_{l_2})\middle|\{c,d\}_m\right)$$

$$\times\,\frac{\hat{\eta}_{\mathrm{a}}}{\eta_{\mathrm{a}}}\,\frac{f_{\hat{a}/A}(\hat{\eta}_{\mathrm{a}},\mu_F^2)}{f_{a/A}(\eta_{\mathrm{a}},\mu_F^2)}\,\frac{\hat{\eta}_{\mathrm{b}}}{\eta_{\mathrm{b}}}\,\frac{f_{\hat{b}/B}(\hat{\eta}_{\mathrm{b}},\mu_F^2)}{f_{b/B}(\eta_{\mathrm{b}},\mu_F^2)}\left(\{\hat{p},\hat{f}\}_{m+1}\middle|\mathcal{R}_{l,k}(z,y,\kappa_\perp)\middle|\{p,f\}_m\right)$$

The phase space is exact after the splitting:

$$d\Gamma^{(m+1)}(\{\hat{p}\}_{m+1};Q)\,\frac{1}{2\hat{p}_{l,1}\cdot\hat{p}_{l,2}} = d\Gamma^{(m)}(\{p\}_m;Q)\,\frac{dy}{y}\,dz\,\frac{d\phi}{2\pi}\,\frac{1-y}{16\pi^2}$$

Splitting Operator

$$
\begin{aligned}
&\left(\{\hat{p},\hat{f},\hat{c},\hat{d}\}_{m+1}\middle|\mathcal{H}(t)\middle|\{p,f,c,d\}_m\right)\\
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\end{aligned}
$$

E.g.: Final state splitting with final state spectator, $q\to q+g$

$$
S_{l,k}(z,y,q,g)=C_{\rm F}\left[\frac{2}{1-z(1-y)}-(1+z)\right]
$$

Splitting Operator

$$
\begin{aligned}
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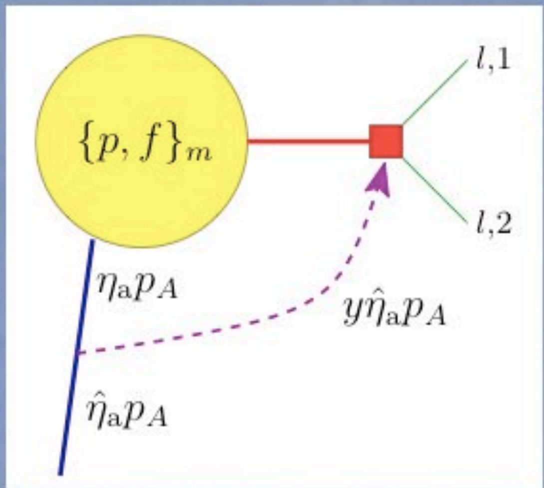

Final state emitter with initial state spectator

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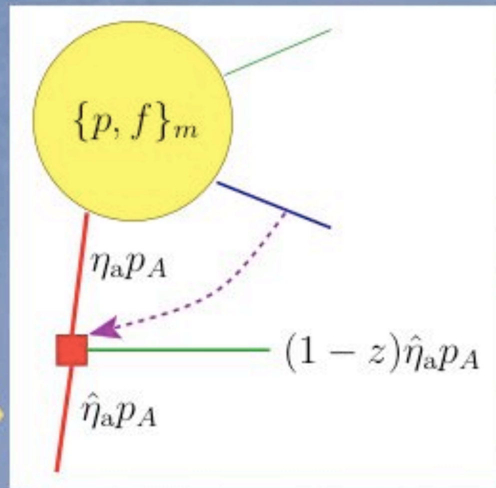

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&\big(\{\hat p,\hat f,\hat c,\hat d\}_{m+1}\big|\mathcal{H}(t)\big|\{p,f,c,d\}_m\big)\\
&\quad=\sum_{\substack{l,k\\k\neq l}}\int_0^1\frac{dy}{y}\int_0^1 dz\int_0^{2\pi}\frac{d\phi}{2\pi}\,\frac{\alpha_s(Q^2e^{-t})}{2\pi}\,\delta\big(t+\log(T_{l,k}(p_l,p_k,z,y)/Q^2)\big)\\
&\qquad\times\frac{\big\langle\{\hat c\}_{m+1}\big|\{\hat d\}_{m+1}\big\rangle}{\big\langle\{c\}_m\big|\{d\}_m\big\rangle}\,\big(\{\hat c,\hat d\}_{m+1}\big|\mathcal{S}_{l,k}(z,y,\hat f_{l_1},\hat f_{l_2})\big|\{c,d\}_m\big)\\
&\qquad\times\frac{\hat\eta_{\rm a}}{\eta_{\rm a}}\frac{f_{\hat a/A}(\hat\eta_{\rm a},\mu_F^2)}{f_{a/A}(\eta_{\rm a},\mu_F^2)}\,\frac{\hat\eta_{\rm b}}{\eta_{\rm b}}\frac{f_{\hat b/B}(\hat\eta_{\rm b},\mu_F^2)}{f_{b/B}(\eta_{\rm b},\mu_F^2)}\,\big(\{\hat p,\hat f\}_{m+1}\big|\mathcal{R}_{l,k}(z,y,\kappa_\perp)\big|\{p,f\}_m\big)
\end{aligned}
$$

$$
\begin{aligned}
&\big(\{\hat p,\hat f\}_{{\rm a,b},m+1}\big|\mathcal{R}_{l,k}(z,y,\kappa_\perp)\big|\{p,f\}_{{\rm a,b},m}\big)=\frac{1}{2}(1-y)\,\delta^{f_l}_{\hat f_{l,1}+\hat f_{l,2}}\,\delta^{a}_{\hat a}\,\delta(\hat\eta_{\rm a}-\eta_{\rm a})\\
&\qquad\times\delta^{b}_{\hat b}\,\delta(\hat\eta_{\rm b}-\eta_{\rm b})\prod_{\substack{i=1\\i\neq l}}^{m}\delta^{f_i}_{\hat f_i}\,\delta^{(4)}(\hat p_k-(1-y)p_k)\prod_{\substack{i=1\\i\neq l,k}}^{m}\delta^{(4)}(\hat p_i-p_i)\\
&\qquad\times\delta^{(4)}\big(\hat p_{l,1}-zp_l-y(1-z)p_k-[2p_l\cdot p_k yz(1-z)]^{1/2}\kappa_\perp\big)\\
&\qquad\times\delta^{(4)}\big(\hat p_{l,2}-(1-z)p_l-yzp_k+[2p_l\cdot p_k yz(1-z)]^{1/2}\kappa_\perp\big)
\end{aligned}
$$

Splitting Operator

Why Catani-Seymour factorization?

Practical reasons:

- Simple and elegant formalism
- Most of the modern NLO computations are based on the dipole method (MCFM, NLOJET++, ...)

More physical reasons:

- The phase space factorization is exact
- It is based on the Sudakov parametrization
- Lorentz covariant formalism
- The dipole factorization is valid in the collinear and soft limit

Shower Cross section

The evolution starts from the simplest configuration, e.g.: $p\bar{p} \to \mathrm{jets}$, the simplest configurations are

$$p\bar{p} \to 2 \text{ partons}$$

The shower cross section is

Starting hard scale

$$\sigma[F] = \big(F\big|D(t_{\mathrm{f}})U(t_{\mathrm{f}}, t_2)\big|\sigma_2\big)$$

Hadronization

Infrared cutoff scale

Matching Born Level Matrix elements to Parton Shower

Outlines:

- Definition of the scheme
- Connection to the slicing method (CKKW method)

Adjoint Splitting Operator

Let us define the operator $\mathcal{H}^\dagger(t)$ according to

$$\big(F\big|\mathcal{H}(t)\Gamma\big|\mathcal{M}_2\big) = \big(\mathcal{M}_2\big|\mathcal{H}^\dagger(t)\Gamma\big|F\big)$$

Since $\mathcal{H}(t)$ always increases the number of partons $\mathcal{H}^\dagger(t)$ always decreases it.

For multiple emission:

$$\begin{aligned}\big(F\big|\mathcal{H}(t_m)\mathcal{H}(t_{m-1})\cdots\mathcal{H}(t_3)\Gamma\big|\mathcal{M}_2\big)\\ = \big(\mathcal{M}_2\big|\mathcal{H}^\dagger(t_3)\cdots\mathcal{H}^\dagger(t_{m-1})\mathcal{H}^\dagger(t_m)\Gamma\big|F\big)\end{aligned}$$

Adjoint Splitting Operator

Let us define the operator $\mathcal{H}^\dagger(t)$ according to

$$\big(F\big|\mathcal{H}(t)\Gamma\big|\mathcal{M}_2\big) = \big(\mathcal{M}_2\big|\mathcal{H}^\dagger(t)\Gamma\big|F\big)$$

$$\begin{aligned}
&\big(\{\tilde{p},\tilde{f},\tilde{c},\tilde{d}\}_m\big|\mathcal{H}^\dagger(t)\big|\{p,f,c,d\}_{m+1}\big) \\
&\quad = \sum_{\substack{i,j\\ \text{pairs}}}\sum_{k\neq i,j}\frac{1}{2p_i\cdot p_j}\,\frac{\eta_{\mathrm{a}}}{\tilde{\eta}_{\mathrm{a}}}\,\frac{\eta_{\mathrm{b}}}{\tilde{\eta}_{\mathrm{b}}}\,\delta\big(t+\log(T_{ij,k}(p_i,p_j,p_k)/Q^2)\big) \\
&\qquad \times \frac{\alpha_s(Q^2e^{-t})}{2\pi}\,\big(\{\tilde{c},\tilde{d}\}_m\big|\boldsymbol{V}_{ij,k}(p_i,p_j,p_k,f_i,f_j)\big|\{c,d\}_{m+1}\big) \\
&\qquad \times \big(\{\tilde{p},\tilde{f}\}_m\big|\mathcal{Q}_{ij,k}\big|\{p,f\}_{m+1}\big)
\end{aligned}$$

Approximated Matrix Element

If $\big|\{p,f,c,d\}_m\big)$ was generated by a shower procedure then the following is a good approximation:

$$\big(\mathcal{M}_m\big|\{p,f,c,d\}_m\big) \approx \underbrace{\int_{t_2}^{t_\mathrm{f}} dt_3 \int_{t_3}^{t_\mathrm{f}} dt_4 \cdots \int_{t_{m-1}}^{t_\mathrm{f}} dt_m \prod_{k=3}^{m} \frac{\alpha_s(\mu_R^2)}{\alpha_s(Q^2 e^{-t_k})} \times \big(\mathcal{M}_2\big|\mathcal{H}^\dagger(t_3)\mathcal{H}^\dagger(t_4)\cdots\mathcal{H}^\dagger(t_m)\big|\{p,f,c,d\}_m\big)}_{\big(\mathcal{A}_m(t_\mathrm{f},t_2)\big|\{p,f,c,d\}_m\big)}$$

$$w_M(\{p,f,c,d\}_m, t_\mathrm{f}, t_2) = \begin{cases} \dfrac{\big(\mathcal{M}_m\big|\{p,f,c,d\}_m\big)}{\big(\mathcal{A}_m(t_\mathrm{f},t_2)\big|\{p,f,c,d\}_m\big)} & \text{if } \mathcal{M}_m \text{ is known} \\ 1 & \text{otherwise} \end{cases}$$

Approximated Matrix Element

If $\big|\{p,f,c,d\}_m\big)$ was generated by a shower procedure then the following is a good approximation:

$$\big(\mathcal{M}_m\big|\{p,f,c,d\}_m\big) \approx \underbrace{\int_{t_2}^{t_\mathrm{f}} dt_3 \int_{t_3}^{t_\mathrm{f}} dt_4 \cdots \int_{t_{m-1}}^{t_\mathrm{f}} dt_m \prod_{k=3}^{m} \frac{\alpha_s(\mu_R^2)}{\alpha_s(Q^2 e^{-t_k})} \times \big(\mathcal{M}_2\big|\mathcal{H}^\dagger(t_3)\mathcal{H}^\dagger(t_4)\cdots\mathcal{H}^\dagger(t_m)\big|\{p,f,c,d\}_m\big)}_{\big(\mathcal{A}_m(t_\mathrm{f},t_2)\big|\{p,f,c,d\}_m\big)}$$

Matrix element reweighting operator:

$$W_M(t_\mathrm{f},t_2) = \sum_m \int \big[d\{p,f,c,d\}_m\big]\,\big|\{p,f,c,d\}_m\big)\big(\{p,f,c,d\}_m\big| \times w_M(\{p,f,c,d\}_m, t_\mathrm{f}, t_2)$$

Matching at Born Level

Expanding the first step of the shower cross section:

$$\big|\sigma(t_{\rm f})\big) = N(t_{\rm f},t_2)\big|\sigma_2\big) + \int_{t_2}^{t_{\rm f}} dt_3\, U(t_{\rm f},t_3)\,\underbrace{\mathcal{H}(t_3)N(t_3,t_2)\big|\sigma_2\big)}_{\sim\,\big|\mathcal{A}_3(t_{\rm f},t_2)\big)}$$

It is better to use the 3-parton matrix element in the second term. Assuming we know $\mathcal{M}_3$

$$\big|\sigma_M(t_{\rm f})\big) = N(t_{\rm f},t_2)\big|\sigma_2\big) + \int_{t_2}^{t_{\rm f}} dt_3\, U(t_{\rm f},t_3) W_M(t_{\rm f},t_2)\mathcal{H}(t_3)N(t_3,t_2)\big|\sigma_2\big)$$

Adding and subtracting the same terms we have

$$\big|\sigma_M(t_{\rm f})\big) = \underbrace{U(t_{\rm f},t_2)\big|\sigma_2\big)}_{\text{Standard shower}} + \int_{t_2}^{t_{\rm f}} dt_3\, U(t_{\rm f},t_3)\underbrace{\big[W_M(t_{\rm f},t_2),\mathcal{H}(t_3)\big]}_{W_M(t_{\rm f},t_2)\mathcal{H}(t_3)-\mathcal{H}(t_3)W_M(t_{\rm f},t_2)} N(t_3,t_2)\big|\sigma_2\big)$$

Matching at Born Level

Expanding the first step of the shower cross section:

$$\big|\sigma(t_\mathrm{f})\big) = N(t_\mathrm{f}, t_2)\big|\sigma_2\big) + \int_{t_2}^{t_\mathrm{f}} dt_3\, U(t_\mathrm{f}, t_3)\, \underbrace{\mathcal{H}(t_3) N(t_3, t_2)\big|\sigma_2\big)}$$

$$[W_M(t_\mathrm{f}, t_2), \mathcal{H}(t_3)]\big|\sigma_2\big) \sim \big|\sigma_3\big) - \mathcal{H}(t_3)\big|\sigma_2\big)$$

It

second term. Assuming we know $\mathcal{M}_3$

$$\big|\sigma_M(t_\mathrm{f})\big) = N(t_\mathrm{f}, t_2)\big|\sigma_2\big) + \int_{t_2}^{t_\mathrm{f}} dt_3\, U(t_\mathrm{f}, t_3) W_M(t_\mathrm{f}, t_2) \mathcal{H}(t_3) N(t_3, t_2)\big|\sigma_2\big)$$

Adding and subtracting the same terms we have

$$\big|\sigma_M(t_\mathrm{f})\big) = \underbrace{U(t_\mathrm{f}, t_2)\big|\sigma_2\big)}_{\text{Standard shower}} + \int_{t_2}^{t_\mathrm{f}} dt_3\, U(t_\mathrm{f}, t_3) \underbrace{[W_M(t_\mathrm{f}, t_2), \mathcal{H}(t_3)]}_{W_M(t_\mathrm{f},t_2)\mathcal{H}(t_3) - \mathcal{H}(t_3)W_M(t_\mathrm{f},t_2)} N(t_3, t_2)\big|\sigma_2\big)$$

Matching at Born Level

Assuming we know $\mathcal{M}_3$, $\mathcal{M}_4, ...,$ $\mathcal{M}_n$, the matched shower cross section is

$$\begin{aligned}
&\big|\sigma_M(t_{\rm f})\big) = N(t_{\rm f},t_2)\big|\sigma_2\big) \\
&\quad + \sum_{m=3}^{n-1}\int_{t_2}^{t_{\rm f}} dt_3 \int_{t_3}^{t_{\rm f}} dt_4 \cdots \int_{t_{m-1}}^{t_{\rm f}} dt_m\, N(t_{\rm f},t_m) W_M(t_{\rm f},t_2)\mathcal{H}(t_m)N(t_m,t_{m-1}) \\
&\qquad\qquad \times \mathcal{H}(t_{m-1})N(t_{m-1},t_{m-2})\cdots\mathcal{H}(t_3)\,N(t_3,t_2)\big|\sigma_2\big) \\
&\quad + \int_{t_2}^{t_{\rm f}} dt_3 \int_{t_3}^{t_{\rm f}} dt_4 \cdots \int_{t_{n-1}}^{t_{\rm f}} dt_n\, U(t_{\rm f},t_n) W_M(t_{\rm f},t_2)\mathcal{H}(t_n)N(t_n,t_{n-1}) \\
&\qquad\qquad \times \mathcal{H}(t_{n-1})N(t_{n-1},t_{n-2})\cdots\mathcal{H}(t_3)\,N(t_3,t_2)\big|\sigma_2\big)
\end{aligned}$$

Matching at Born Level

After some algebraic manipulation:

$$\big|\sigma_M(t_\mathrm{f})\big) = \big|\sigma_\Delta(t_\mathrm{f})\big) \equiv N(t_\mathrm{f},t_2)\big|\sigma_2\big)$$
$$+\sum_{m=3}^{n-1}\int_{t_2}^{t_\mathrm{f}} dt_m\, N(t_\mathrm{f},t_m)W_\Delta(t_\mathrm{f},t_m,t_2)\big|\sigma_m\big)$$
$$+\int_{t_2}^{t_\mathrm{f}} dt_n\, U(t_\mathrm{f},t_n)W_\Delta(t_\mathrm{f},t_n,t_2)\big|\sigma_n\big)$$

$$W_\Delta(t_\mathrm{f},t,t_2) = \sum_m \int \big[d\{p,f,c,d\}_m\big]\,\big|\{p,f,c,d\}_m\big)\big(\{p,f,c,d\}_m\big|$$
$$\times \lim_{\delta\to 0}\int_{t_2}^{t} dt_{m-1}\int_{t_2}^{t_{m-1}} dt_{m-2}\cdots\int_{t_2}^{t_4} dt_3$$
$$\times \frac{\big(\mathcal{M}_2\big|N(t_3,t_2)\mathcal{H}^\dagger(t_3)\cdots N(t,t_{m-1})\mathcal{H}^\dagger(t)\big|\{p,f,c,d\}_m\big)}{\big(\mathcal{A}_m(t_\mathrm{f},t_2)\big|\{p,f,c,d\}_m\big)+\delta}$$

Slicing Method

(Catani-Krauss-Kuhn-Webber method)

Defining the matching scale $t_\mathrm{f} > t_\mathrm{ini} > t_2$ and using the group decomposation property:

$$\big|\sigma(t_\mathrm{f})\big) = U(t_\mathrm{f}, t_\mathrm{ini}) U(t_\mathrm{ini}, t_2)\big|\sigma_2(t_2)\big) \approx U(t_\mathrm{f}, t_\mathrm{ini})\big|\sigma_\Delta(t_\mathrm{ini})\big)$$

The CKKW method use a simplified Sudakov rewighting operator based on the $k_\perp$ jet algorithm

$$\begin{aligned}\big|\sigma_\mathrm{CKKW}(t_\mathrm{f})\big) = {} & U(t_\mathrm{f}, t_\mathrm{ini}) N(t_\mathrm{ini}, t_2)\big|\sigma_2\big) \\ & + \sum_{m=3}^{n-1} \int_{t_2}^{t_\mathrm{ini}} dt_m\, U(t_\mathrm{f}, t_\mathrm{ini}) N(t_\mathrm{ini}, t_m) W_\mathrm{CKKW}(t_\mathrm{ini}, t_m, t_2)\big|\sigma_m\big) \\ & + \int_{t_2}^{t_\mathrm{ini}} dt_n\, U(t_\mathrm{ini}, t_n) W_\mathrm{CKKW}(t_\mathrm{ini}, t_n, t_2)\big|\sigma_n\big)\end{aligned}$$

Matching Parton Shower and NLO Computation

Parton Shower at NLO

Let us calculate the N-jet cross section. The matrix element improved cross section is

$$\begin{aligned}\big(F_N\big|\sigma_\Delta(t_\mathrm{f})\big) = &\int_{t_2}^{t_\mathrm{f}} dt_N\, \big(F_N\big|N(t_\mathrm{f},t_N)W_\Delta(t_\mathrm{f},t_N,t_2)\big|\sigma_N\big) \\ &+ \int_{t_2}^{t_\mathrm{f}} dt_{N+1}\, \big(F_N\big|U(t_\mathrm{f},t_{N+1})W_\Delta(t_\mathrm{f},t_{N+1},t_2)\big|\sigma_{N+1}\big)\end{aligned}$$

Expanding it in α_s then we have

$$\big(F_N\big|\sigma_\Delta\big) = \int_N d\sigma^B\left(1 + \frac{\alpha_s}{2\pi}W_\Delta^{(1)}\right) + \underbrace{\int_{N+1}\left[d\sigma^R - d\sigma^A\right]}_{\text{Real - Dipoles}} + \mathcal{O}(\alpha_s^2)$$

Born term

"Quasi virtual"

Matching at NLO Level

The NLO parton shower for an N-jet cross section is

$$
\begin{aligned}
\big(F_N\big|\sigma_{\mathrm{NLO}}(t_{\mathrm{f}})\big) = & \int_{t_2}^{t_{\mathrm{f}}} dt_N \, \big(F_N\big|N(t_{\mathrm{f}},t_N)W_\Delta(t_{\mathrm{f}},t_N,t_2)\big|\sigma_N\big) \\
& + \int_{t_2}^{t_{\mathrm{f}}} dt_{N+1} \, \big(F_N\big|U(t_{\mathrm{f}},t_{N+1})W_\Delta(t_{\mathrm{f}},t_{N+1},t_2)\big|\sigma_{N+1}\big) \\
& + \int_{t_2}^{t_{\mathrm{f}}} dt_N \, \big(F_N\big|U(t_{\mathrm{f}},t_N)W_\Delta(t_{\mathrm{f}},t_N,t_2)\big|\sigma_N^{(1)}\big)
\end{aligned}
$$

$$\sim |\mathcal{M}_N|^2_{1-\mathrm{loop}} + \boldsymbol{I} \otimes |\mathcal{M}_N|^2$$

$$\big|\sigma_N^{(1)}\big) = -\frac{\alpha_s}{2\pi} W_\Delta^{(1)} \big|\sigma_N\big) + \big|\sigma_N^{I+V}\big) + \big|\sigma_N^{P+K}\big)$$

$$\sim (\boldsymbol{P}(\mu_F) + \boldsymbol{K}) \otimes |\mathcal{M}_N|^2$$

Summary (hep-ph/0601021)

- We defined a new formalism for describing the parton shower
- Exact kinematics, Lorentz invariant and covariant formalism, improved soft gluon
- No phase space cut parametes at all, only the infrared cutoff parameter
- Clear way to add higher order to the shower
- It is possible to add massive fermions in the same way
- Matched to the LO matrix elements
- Matched to the "NLO matrix elements"