Gravitational radiation: Lecture 1

Alessandra Buonanno

Laboratoire “AstroParticule et Cosmologie” (APC), Paris
Lectures’ content

Lecture 1:

- Einstein equations for weak gravitational fields
- Propagation of GWs in vacuum: plane wave solution
- Interaction of GWs with free-falling particles
- Local Lorentz frame versus transverse traceless frame
- GWs carry off energy and angular-momentum
Lectures’ content

Lecture 2:

- Quadrupolar wave generation in linearized Einstein theory

- Brief survey of GW sources:
  - Black hole and/or neutron star binaries
  - pulsars
  - supernovae
  - stochastic background
References

**Landau-Lifshitz:** *Field Theory*, Chap. 11, 13

**Schutz:** *A first course in general relativity*, Chap. 8, 9

**Weinberg:** *Gravitation and Cosmology*, Chap. 7, 10

**Misner-Thorne-Wheeler:** *Gravitation*, Chap. 8

**Course by Thorne available on the web:** Lectures 4, 5 & 6
Relativistic units:

\[ G = 1 = c \quad \Rightarrow \quad \text{Mass, space and time have same units} \]

\[ 1 \text{ sec} \sim 3 \times 10^{10} \text{ cm} \]

\[ 1 M_\odot \sim 5 \times 10^{-6} \text{ sec} \]
A bit of gravitational-wave history

- **In 1916 Einstein realized the propagation effects at finite velocity in the gravitational equations and predicted the existence of wave-like solutions of the linearized vacuum field equations**

- **Works by Eddington, Einstein et al. in the 20-30s trying to understand whether the radiative degrees of freedom were physical**
  - Complications and subtleties: Non-linearities and invariance under coordinate transformations

- **The work by Bondi in the mid 50s, applied to self-gravitating systems like binaries made of neutron stars and/or black holes, proved that gravitational waves carry off energy and angular-momentum**
Brief summary of Einstein equations and notations

\[ S = S_g + S_{\text{matter}} \]

\[ S_g = \frac{1}{16\pi G} \int d^4 x \sqrt{-g} \ R \quad - \quad \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} = T_{\mu\nu} \]

\[ \eta_{\mu\nu} = (-, +, +, +) \quad \text{with} \quad \mu, \nu = 0, 1, 2, 3 \quad \text{and} \quad i, j = 1, 2, 3 \]

by imposing the principle of minimal action

\[ \int (G_{\mu\nu} - \frac{8\pi G}{c^4} T_{\mu\nu}) \delta g^{\mu\nu} \sqrt{-g} = 0 \]

\[ \Rightarrow G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \]
Brief summary of Einstein equations [continued]

\[ R_{\mu \nu \rho \sigma} = \frac{\partial \Gamma_{\mu \rho}^{\nu}}{\partial x^{\sigma}} - \frac{\partial \Gamma_{\mu \sigma}^{\nu}}{\partial x^{\rho}} + \Gamma_{\lambda \rho}^{\nu} \Gamma_{\mu \sigma}^{\lambda} - \Gamma_{\lambda \sigma}^{\nu} \Gamma_{\mu \rho}^{\lambda} \]

\[ \Gamma_{\nu \rho}^{\mu} = \frac{1}{2} g^{\mu \lambda} \left( \frac{\partial g_{\lambda \nu}}{\partial x^{\rho}} + \frac{\partial g_{\lambda \rho}}{\partial x^{\nu}} - \frac{\partial g_{\rho \nu}}{\partial x^{\lambda}} \right) \]

more explicitly:

\[ R_{\mu \nu \rho \sigma} = \frac{1}{2} \left( \frac{\partial^2 g_{\mu \sigma}}{\partial x^{\nu} \partial x^{\rho}} + \frac{\partial^2 g_{\nu \rho}}{\partial x^{\mu} \partial x^{\sigma}} - \frac{\partial^2 g_{\mu \rho}}{\partial x^{\nu} \partial x^{\sigma}} - \frac{\partial^2 g_{\nu \sigma}}{\partial x^{\mu} \partial x^{\rho}} \right) \]

\[ + \frac{1}{2} g_{\lambda \alpha} \left( \Gamma_{\nu \rho}^{\lambda} \Gamma_{\mu \sigma}^{\alpha} - \Gamma_{\nu \sigma}^{\lambda} \Gamma_{\mu \rho}^{\alpha} \right) \]

Bianchi identity: \[ R_{\mu \nu \rho ; \sigma}^{\lambda} + R_{\mu \sigma \nu ; \rho}^{\lambda} + R_{\mu \rho \sigma ; \nu}^{\lambda} = 0 \]
Brief summary of Einstein equations [continued]

Ricci tensor: \[ R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu} \]

Scalar tensor: \[ R = g^{\mu\nu} R_{\mu\nu} \]

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \]

- Non-linear equations with well-posed initial value structure

- \(4 \times 4 = 16\) differential equations, but \(G_{\mu\nu}\) and \(T_{\mu\nu}\) are symmetric tensors \(\Rightarrow 10\) differential equations, but because of Bianchi identity \(G_{\mu\nu}^{;\nu} = 0 \Rightarrow 6\) differential equations to be solved when \(T_{\mu\nu}\) is given
Einstein equations for weak gravitational fields in flat spacetime

\[ g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu} \quad |h_{\mu \nu}| \ll 1 \]

\[ \Gamma^\mu_{\nu \rho} = \frac{1}{2} \left( \partial_\rho h^\mu_{\nu} + \partial_\nu h^\mu_{\rho} - \partial^\mu h_{\nu \rho} \right) + \mathcal{O}(|h|^2) \]

\[ R_{\mu \nu \rho \sigma} = \eta_{\mu \alpha} \partial_\rho \Gamma^\alpha_{\nu \sigma} - \eta_{\mu \alpha} \partial_\sigma \Gamma^\alpha_{\nu \rho} + \mathcal{O}(|h|^2) \]

\[ = \frac{1}{2} \left( h_{\mu \sigma, \nu \rho} + h_{\nu \rho, \mu \sigma} - h_{\mu \rho, \nu \sigma} - h_{\nu \sigma, \mu \rho} \right) + \mathcal{O}(|h|^2) \]

\[ G_{\nu \sigma} = R_{\nu \sigma} - \frac{1}{2} \eta_{\nu \sigma} R = \]

\[ \frac{1}{2} \left[ h_{\mu \sigma, \nu} + h_{\mu \nu, \sigma} - h_{, \nu \sigma} - h_{\nu \sigma, \mu} - \eta_{\nu \sigma} h_{\mu \alpha,} \right. \left. \alpha^\mu + \eta_{\nu \sigma} h_{, \alpha} \alpha + \mathcal{O}(|h|^2) \right] \]
Trace reverse tensor $\bar{h}_{\mu\nu}$

At linear order we can write: $h^\mu_{\beta} = \eta^{\mu\alpha} h_{\alpha\beta}$, $h = \eta^{\alpha\beta} h_{\alpha\beta}$

Introducing $\bar{h}^{\alpha\beta} = h^{\alpha\beta} - \frac{1}{2} \eta^{\alpha\beta} h$ (note that $\bar{h} = -h$)

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R \simeq -\frac{1}{2} \left[ \bar{h}_{\mu\nu,\lambda} + \eta_{\mu\nu} \bar{h}_{\lambda\rho,\lambda} - 2 \bar{h}_{\mu\lambda,\nu} + \mathcal{O}(|\bar{h}|^2) \right]$$

Imposing the Lorenz gauge (or harmonic or De Donder gauge) $\bar{h}_{\mu\nu,\nu} = 0$

Einstein field equations in linearized theory

$$G_{\mu\nu} \simeq -\frac{1}{2} \bar{h}_{\mu\nu,\lambda} = \frac{8\pi G}{c^4} T_{\mu\nu} \Rightarrow \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$
Lorenz gauge can always be imposed

\[ x_{\text{new}}^\mu = x^\mu + \xi^\mu(x) \] with \( \xi^\mu \) an arbitrary and infinitesimal vector field

\[ g_{\mu\nu}^{\text{new}} = g_{\mu\nu} - \xi_{\mu\nu} - \xi_{\nu\mu} \]

\[ \bar{h}_{\mu\nu}^{\text{new}} = \bar{h}_{\mu\nu} - \eta^{\mu\rho} \xi_{\rho,\nu} - \eta^{\lambda\nu} \xi_{\mu,\lambda} + \eta_{\mu\nu} \xi_{\rho,\rho} \]

\[ \bar{h}_{\text{new},\mu\nu} = \bar{h}_{\mu\nu} - \eta^{\lambda\nu} \xi_{\mu,\lambda} = 0 \Rightarrow \Box \xi^\mu = \bar{h}_{\mu\nu}^{\rho,\nu} \]

• \( \xi^\mu \) exists for any well behaved \( \bar{h}_{\mu\nu} \)

• \( \xi^\mu \) is not unique, we can always add to it \( q^\mu \) such that \( \Box q^\mu = 0 \)
Propagation of GWs in vacuum

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \bar{h}^{\mu\nu} = 0 \quad \text{with} \quad \partial_\nu \bar{h}^{\mu\nu} = 0
\]

- Plane wave solution:

\[
\bar{h}^{\mu\nu} = A \epsilon^{\mu\nu}(k) e^{ik_\alpha x^\alpha} \quad \text{with} \quad k_\nu \epsilon^{\mu\nu} = 0 \quad \epsilon^{\mu\nu} \rightarrow \text{polarization tensor}
\]

- General solution:

\[
\bar{h}^{\mu\nu} = \text{Re} \left[ \int d^3k \, A^{\mu\nu}(k) e^{ik_\alpha x^\alpha} \right] \quad \text{with} \quad k^\mu = (\omega, \vec{k}) \quad \text{and} \quad k^\mu A_{\mu\nu} = 0
\]

Using the freedom within Lorenz gauge \( \Rightarrow \) we can determine the only physical radiative components in \( \bar{h}^{\mu\nu} \Rightarrow \bar{h}^{\mu\nu}_{TT} \)
Imposing transverse-traceless gauge

We choose \( q_\mu = B_\mu e^{i k_\alpha x^\alpha} \) with \( k_\alpha k^\alpha = 0 \) \( (\Box q_\mu = 0) \)

\[
A_{\mu\nu} = A_{\mu\nu}^{\text{old}} - iB_\mu k_\nu - iB_\nu k_\mu + i\eta_{\mu\nu} B^\rho k_\rho
\]

We impose:

1. \( A_{\mu\nu} k^\nu = 0 \)
2. \( A_{\mu\nu} \eta^{\mu\nu} = 0 \)
3. If \( U^\nu \) is a constant timelike unit vector \( (U_\nu U^\nu = -1) \) we impose \( A_{\mu\nu} U^\mu = 0 \)

This set of equations determine \( B_\mu \)
Imposing transverse-traceless gauge [continued]

1. Choosing $U^\mu = \delta^\mu_0 \Rightarrow A_{\mu 0} = 0 \Rightarrow A_{00} = A_{x0} = A_{y0} = A_{z0} = 0$

assuming the wave travels along $z$

2. $A_{\mu \nu} k^\nu = 0 \Rightarrow A_{\mu z} = 0 \Rightarrow A_{0z} = A_{xz} = A_{yz} = A_{zz} = 0$

3. $A_{\mu \nu} \eta^{\mu \nu} = 0 \Rightarrow A_{xx} = -A_{yy}$

$$A_{\mu \nu}^{TT} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Only two independent components: $A_{xx}$ and $A_{xy}$
Linearly polarized waves in EM and GW theory

- In EM theory linearly polarized vectors are:

\[ \mathbf{e}_x \text{ and } \mathbf{e}_y \]

- In GW theory linearly polarized tensors are:

\[
\begin{align*}
\mathbf{e}_+ &= \mathbf{e}_x \times \mathbf{e}_x - \mathbf{e}_y \times \mathbf{e}_y \\
\mathbf{e}_\times &= \mathbf{e}_x \times \mathbf{e}_y + \mathbf{e}_y \times \mathbf{e}_x
\end{align*}
\]

\[
(u \times v)(\lambda, \mathbf{q}) = (\lambda \cdot u)(\mathbf{q} \cdot \mathbf{v})
\]

\[
\begin{align*}
\mathbf{e}_+ &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \\
\mathbf{e}_\times &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
Circularly polarized waves in EM and GW theory

- In EM theory circularly polarized vectors are:

\[ e_R = \frac{1}{\sqrt{2}}(e_x + i e_y) \quad \text{and} \quad e_L = \frac{1}{\sqrt{2}}(e_x - i e_y) \]

- In GW theory circularly polarized tensors are:

\[ e_R = \frac{1}{\sqrt{2}}(e_+ + i e_\times) \quad \text{and} \quad e_L = \frac{1}{\sqrt{2}}(e_+ - i e_\times) \]

\[ e_+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad e_\times = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]
GWs have helicity 2

Any plane wave $\psi$ which is transformed by a rotation of any angle $\theta$ around the direction of propagation into $\psi' = e^{i h \theta} \psi$ is said to have helicity $h$.

Let us rotate the coordinate system around $z$ by $\theta$

$$x' = x \cos \theta + y \sin \theta \quad y' = y \cos \theta - x \sin \theta$$

$$e_{x'} = e_x \cos \theta + e_y \sin \theta \quad e_{y'} = -e_x \sin \theta + e_y \cos \theta$$

$$e_{+'} = e_+ \cos 2\theta + e_\times \sin 2\theta \quad e_{\times'} = -e_+ \sin 2\theta + e_\times \cos 2\theta$$
Newtonian description of tidal gravity

- Two point particles A and B falling freely under the action of external Newtonian potential $\Phi$

- A and B at time $t = 0$ are separated by small distance $\xi$ and have equal velocity $v_A(0) = v_B(0)$

- For $t > 0$, A and B experiences slightly different gravitational potential and accelerations $g = -\nabla \Phi$

\[
\begin{align*}
\dot{\xi}^i &= x_A^i - x_B^i \\
\ddot{\xi}^i &= x_A^{\ddot{i}} - x_B^{\ddot{i}} = -(\frac{\partial \Phi}{\partial x^i})_B + (\frac{\partial \Phi}{\partial x^i})_A \approx -\left(\frac{\partial^2 \Phi}{\partial x^i \partial x^j}\right) \xi_j \\
\epsilon_{ij} &= -\left(\frac{\partial \Phi}{\partial x^i \partial x^j}\right) \Rightarrow \text{Newtonian tidal gravitational field}
\end{align*}
\]
Equation of geodesic deviation

Pair of nearby freely-falling particles A and B traveling on trajectories $x^\mu(\tau)$ and $x^\mu(\tau) + \xi^\mu$

$$0 = \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau}$$

$$0 = \frac{d^2(x^\mu + \xi^\mu)}{d\tau^2} + \Gamma^\mu_{\nu\lambda}(x + \xi) \frac{d(x^\nu + \xi^\nu)}{d\tau} \frac{d(x^\lambda + \xi^\lambda)}{d\tau}$$

Taking the difference and limiting to first order in $\xi$

$$\nabla_U \nabla_U \xi^\lambda = R^\lambda_{\nu\mu\rho} \xi^\mu U^\nu U^\rho \quad U^\alpha = \frac{dx^\alpha}{d\tau}$$
Interaction of GWs with free-falling particles using local Lorentz frame

- Two test particles A and B initially at rest one respect to the other in absence of GWs
- Local Lorentz frame attached to particle A, with spatial origin at $x^j = 0$ and coordinate time equal to proper time $x^0 = t$
- By definition of LLF, the metric $g_{\mu\nu}$ of a LLF observer reduces to Minkowski metric at the origin and all its first derivatives must vanish at the origin

$$ds^2 = -dt^2 + dx^2 + \mathcal{O} \left( \frac{|x|^2}{\mathcal{R}} \right)$$

$\mathcal{R}$ being the curvature radius: $\mathcal{R}^{-2} = |R_{\mu\nu\rho\sigma}|$
Geodesic deviation equation in LLF

\[ \nabla_U \nabla_U \xi^\alpha = R^\alpha_{\nu\lambda\rho} \frac{dx^\nu}{dt} \frac{dx^\rho}{dt} \]

\[ \nabla_U \nabla_U \xi^\alpha = U^\beta \nabla_\beta (U^\lambda \nabla_\lambda \xi^\alpha) = U^\beta U^\lambda \nabla_\beta (\xi^\alpha,_{\lambda} + \Gamma^\alpha_{\lambda\sigma} \xi^\sigma) \]

In the LLF of particle A:
\[ \Gamma^\sigma_{\alpha\beta} = 0 \quad \Rightarrow \quad \nabla_U \nabla_U \xi^\alpha = U^\beta U^\lambda (\xi^\alpha,_{\lambda\beta} + \Gamma^\alpha_{\lambda\sigma,\beta} \xi^\sigma) \]

\[ U^\alpha = \delta^\alpha_0 \quad \Rightarrow \quad \nabla_U \nabla_U \xi^\alpha = \ddot{\xi}^\alpha + \Gamma^\alpha_{0\sigma,0} \xi^\sigma \]

\[ \dot{g}_{\mu\nu} \sim O \left( \frac{|x|^2}{\mathcal{R}} \right) \quad \Rightarrow \quad \nabla_U \nabla_U \xi^\alpha = \ddot{\xi}^\alpha \]

Assuming \( \xi^0 = 0 \quad \Rightarrow \quad \frac{d^2 \xi^i}{dt^2} = R^i_{0i0} \xi^i \)
Geodesic deviation equation in LLF [continued]

If \( \bar{x}^\mu \) and \( \bar{g}^{\mu\nu} \) refer to TT gauge (\( h_{\times}^{\text{TT}} = 0, h_{+}^{\text{TT}} \neq 0 \)):

\[
\bar{g}^{\mu\nu} = \eta_{\mu\nu} + \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & h_{\times}^{\text{TT}} & 0 & 0 \\
0 & 0 & -h_{+}^{\text{TT}} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

The local Lorentz metric \( g_{\mu\nu} \) should reduce to Minkowski metric at the origin and all its first derivatives must vanish

\[
\bar{t} = t - \dot{h}_{\times}^{\text{TT}} (x^2 - y^2)/4 \\
\bar{x} = x - h_{\times}^{\text{TT}} x/2 \\
\bar{y} = y + h_{+}^{\text{TT}} y/2 \\
\bar{z} = z + \dot{h}_{+}^{\text{TT}} (x^2 - y^2)/4
\]

\[
g^{\mu\nu} = \eta_{\mu\nu} - 2 \begin{pmatrix}
\Phi(t) & 0 & 0 & \Phi(t) \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\Phi(t) & 0 & 0 & \Phi(t)
\end{pmatrix}
\]

\[
\Phi(t) = -\frac{1}{4} \dddot{h}_{\times}^{\text{TT}} (x^2 - y^2) \implies \frac{d^2 \xi^j}{dt^2} = -\frac{\partial^2 \Phi}{\partial x^j \partial x^k} \xi^k
\]
Geodesic deviation equation in LLF [continued]

\[ R^i_{0j0} = \frac{\partial x^i}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^0} \frac{\partial x^\lambda}{\partial x^j} \frac{\partial x^\rho}{\partial x^0} R^{TT\mu}_{\nu\lambda\rho} \approx R^{TTi}_{0j0} \Rightarrow R^i_{0j0} = \frac{1}{2} \ddot{h}^{TT}_{ij} \]

\[ \frac{d^2 \xi_j}{dt^2} = R^j_{0i0} \xi^i = \frac{1}{2} \ddot{h}^{TT}_{ij} \xi^j(0) \Rightarrow \delta \xi^i = \frac{1}{2} h^{TT}_{ij} \xi^j(0) \]

- The acceleration of particle B in the LLF of particle A is: \( a^i = \frac{F^i}{m_B} = \frac{1}{2} \ddot{h}^{TT}_{ij} \xi^j(0) \)

- The observer in LLF of particle A concludes that particle B is subjected to the force \( \mathbf{F} \), whereas for an observer in LLF of particle B, particle B is just free falling.

**In GW interferometers:** A → mirror at beam splitter; B → mirror at end of arm cavity

\[ \frac{\delta \xi}{\xi(0)} = \frac{\delta \xi}{L} = h \quad \text{If} \quad L \sim 3 \text{ km, } h \sim 10^{-21} \Rightarrow \delta \xi \sim 10^{-16} \text{ m!} \]
Geodesic deviation equation in LLF [continued]

The LLF is useful to do calculations as long as we can use the metric in the form $g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}\left(\frac{x^2}{\mathcal{R}^2}\right)$, i.e., as long as we can disregard $x^2$ corrections

$$\mathcal{R}^{-2} = |R_{\mu\nu\rho\sigma}| \sim |R_{i0j0}| \sim \ddot{h} \sim \frac{h}{\lambda_{GW}^2}$$

$$\frac{x^2}{\mathcal{R}^2} \sim \frac{L^2 h}{\lambda_{GW}^2} \ll 1 \quad \text{if} \quad L \ll \lambda_{GW}$$

- Ground-based detectors (LIGO, VIRGO, TAMA, GEO): $L \ll \lambda_{GW} \sim 10^3$ km

- Space-based detectors (LISA): $L \sim \lambda_{GW} \sim 5 \times 10^6$ km
Interaction of GWs with free-falling particles using TT gauge

- Two test particles A and B initially at rest one respect to the other in absence of GWs

- \( U^\alpha \) being the 4-velocity of particle A

\[
\frac{dU^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu} U^\mu U^\nu = 0
\]

\[
\Gamma^\alpha_{\mu\nu} = \frac{1}{2} \eta^{\alpha\beta} (h_{\beta\mu,\nu}^{TT} + h_{\nu\beta,\mu}^{TT} - h_{\mu\nu,\beta}^{TT})
\]

- Initially \( U^\alpha = \delta_0^\alpha \) \( \Rightarrow \)

\[
\frac{dU^\alpha}{d\tau} = -\Gamma_0^\alpha = -\frac{1}{2} \eta^{\alpha\beta} (h_{0\beta,0}^{TT} + h_{\beta0,0}^{TT} - h_{00,\beta}^{TT}) = 0!
\]

The particles A and B do not move!
Equivalence between TT gauge and local Lorentz gauge

Proper distance in the two frames (assume that A and B are along $x$-axis and only $h_+ \neq 0$)

- **LLF:**
  \[
  (\Delta s)^2 = g_{xx} (\Delta x)^2
  \]

  \[g_{xx} = 1 \quad \text{but} \quad (\Delta x)^2 = (L + \frac{1}{2}h_+ L)^2\]

  \[\Rightarrow \Delta s = L \left(1 + \frac{1}{2}h_+\right)\]

- **TTF:**
  \[
  (\Delta s)^2 = g_{xx} (\Delta x)^2
  \]

  \[g_{xx} = 1 + h_+ \quad \text{but} \quad (\Delta x)^2 = L^2\]

  \[\Rightarrow \Delta s = L \left(1 + \frac{1}{2}h_+\right)\]

Proper distances are the same!
**Interaction between GW and ring of free-falling particles:** $h_{TT}^T$

GW propagating along $z$-axis

Case: $h_{xx}^{TT} = -h_{yy}^{TT} \equiv h_{+}^{TT} \neq 0 \quad h_{xy}^{TT} = h_{yx}^{TT} \equiv h_{\times}^{TT} = 0$

\[
\delta \xi_x = +\frac{1}{2} h_{xx}^{TT} \xi_x(0)
\]

\[
\delta \xi_y = -\frac{1}{2} h_{xx}^{TT} \xi_y(0)
\]
**Interaction between GW and ring of free-falling particles:** \( h_{\times}^{TT} \)

GW propagating along \( z \)-axis

Case: \( h_{xx}^{TT} = -h_{yy}^{TT} \equiv h_{++}^{TT} = 0 \quad h_{xy}^{TT} = h_{yx}^{TT} \equiv h_{\times}^{TT} \neq 0 \)

\[
\delta \xi_x = +\frac{1}{2} h_{xy}^{TT} \xi_y(0)
\]

\[
\delta \xi_y = +\frac{1}{2} h_{xy}^{TT} \xi_x(0)
\]
Lines of force for $h_{+T}^T$ and $h_{\times T}^T$
Energy-momentum pseudo-tensor

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with} \quad |h_{\mu\nu}| \ll 1 \]

\[ R^{(1)}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R^{(1)} = \frac{8\pi G}{c^4} (T_{\mu\nu} + \tau_{\mu\nu}) \quad R^{(1)}_{\mu\nu} \sim [\partial \partial h]_{\mu\nu} \]

\[ \tau_{\mu\nu} = \frac{c^4}{8\pi G} \left( R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R - R^{(1)}_{\mu\nu} + \frac{1}{2} \eta_{\mu\nu} R^{(1)} \right) \sim [\partial h \partial h]_{\mu\nu} + \cdots \]

\[ t^{\nu\lambda} = \eta^{\nu\mu} \eta^{\lambda\alpha} \left( \underbrace{T_{\mu\alpha}}_{\text{matter's EMT}} + \underbrace{\tau_{\mu\alpha}}_{\text{grav field EMPT}} \right) \]

\[ t^{\mu\nu} \text{ is locally conserved} \Rightarrow t^{\mu\nu},_{\nu} = 0 \]

[matter EMT satisfies \( T^{\mu\nu},_{\nu} = 0 \)]
GWs carry off energy and angular momentum

\[ t^{\mu \nu} = 0 \quad \Rightarrow \quad \text{for any finite system of volume } V \text{ bounded by a surface } S \]

\[ \Rightarrow \quad P^\lambda = \frac{d}{dt} \int_V t^{0 \lambda} \, d^3x = - \int_S t^{i \lambda} \, n_i \, dS \]

\[ P^\lambda \quad \Rightarrow \quad \text{EM vector of matter and gravitational field; } \quad t^{i \lambda} \quad \Rightarrow \quad \text{flux} \]

\[ \lambda = 0 \quad \Rightarrow \quad \frac{dE}{dt} = - \int_S t^{i 0} \, n_i \, dS = - \int \tau^{i 0} \, n_i \, dS \]

For a plane GW propagating along \( z \)-axis, oscillating at frequency \( f_{GW} \):

\[ c < \tau^{0z} > = \frac{c^5}{16 \pi G} < \frac{1}{2} \dot{h}^2_{xx} + \frac{1}{2} \dot{h}^2_{yy} + \dot{h}^2_{xy} > \sim \frac{c^3}{G} f^2_{GW} h^2 \]

Supernova at 20 kpc: \( c < \tau^{0z} > \sim 400 \frac{\text{erg}}{\text{cm}^2 \text{sec}} \left( \frac{f_{GW}}{1 \text{kHz}} \right)^2 \left( \frac{h}{10^{-21}} \right)^2 \)
Comparison with other kind of radiation from supernovae

- From neutrino $\sim 10^5 \frac{\text{erg}}{\text{cm}^2 \text{sec}}$ during 10 secs
- From optical radiation $\sim 10^{-4} \frac{\text{erg}}{\text{cm}^2 \text{sec}}$ during one week