Gravitational radiation: Lecture 2

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Lectures' content

Lecture 2:

- **Quadrupolar wave generation in linearized Einstein theory**
- **Brief survey of GW sources:**
  - **binaries made of black holes and/or neutron stars**
  - **pulsars**
  - **supernovae**
  - **stochastic background**

Quadrupole nature of GW emission [naive way]

EM theory: Luminosity $\propto \ddot{\mathbf{d}}^2$ $\quad \mathbf{d} = e\,\mathbf{x} \Rightarrow$ electric dipole moment

- GW theory: electric dipole moment $\Rightarrow$ mass dipole moment

$$\mathbf{d} = \sum_i m_i\,\mathbf{x}_i \Rightarrow \dot{\mathbf{d}} = \sum_i m_i\,\dot{\mathbf{x}}_i = \mathbf{P}$$

Conservation of momentum $\Rightarrow$ no mass dipole radiation exists in GR

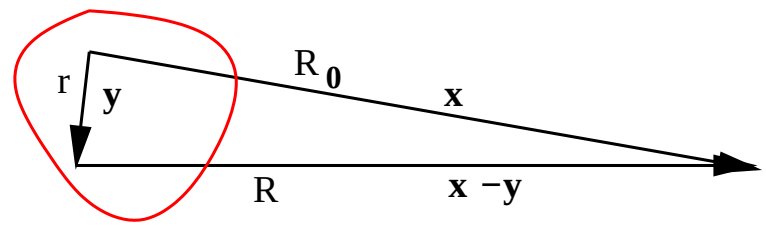
- GW theory: magnetic dipole moment $\Rightarrow$ current dipole moment

$$\mu = \sum_i m_i\,\mathbf{x}_i \times \dot{\mathbf{x}}_i = \mathbf{J}$$

Conservation of angular momentum $\Rightarrow$ no current dipole radiation exists in GR

Quadrupolar wave generation in linearized theory

$$\Box \bar{h}^{\mu\nu} = -\frac{16\pi G}{c^4} T^{\mu\nu} \qquad \partial_\nu \bar{h}^{\mu\nu} = 0 \qquad \Rightarrow$$



like retarded potentials in EM:

$$\bar{h}_{\mu\nu}(\mathbf{x}) = \frac{4G}{c^4} \int T_{\mu\nu}(\mathbf{y}, t - \frac{R}{c}) \frac{d^3y}{|\mathbf{x}-\mathbf{y}|}$$

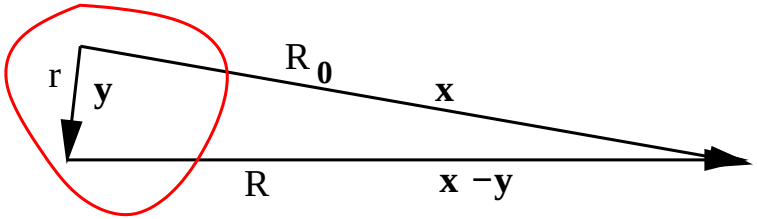
$$R = |\mathbf{x} - \mathbf{y}| = \sqrt{r^2 + R_0^2 - 2\mathbf{r}\cdot\mathbf{R}_0} = R_0 \sqrt{1 - \frac{2\mathbf{n}\cdot\mathbf{y}}{R_0} + \frac{r^2}{R_0^2}}$$

expanding in $\mathbf{y}/R_0 \quad \Rightarrow \quad R \simeq R_0 \left(1 - \frac{\mathbf{n}\cdot\mathbf{y}}{R_0}\right) = R_0 - \mathbf{n}\cdot\mathbf{y}$

$$\bar{h}_{\mu\nu} \simeq \frac{4G}{c^4} \frac{1}{R_0} \int T_{\mu\nu}(\mathbf{y}, t - \frac{R_0}{c} + \frac{\mathbf{n}\cdot\mathbf{y}}{c})\, d^3y$$

Quadrupolar wave generation in linearized theory [continued]

- $\frac{\mathbf{n}\cdot\mathbf{y}}{c}$ can be neglected if source mass distribution doesn't vary much during this time.



If T typical time of variation of source

$$\Rightarrow \quad \frac{\mathbf{n}\cdot\mathbf{y}}{c} \sim \frac{a}{c} \ll T \sim \frac{\lambda_{GW}}{c} \quad \Rightarrow \quad \lambda_{\rm GW} \gg a$$

- Since $T \sim \frac{a}{v} \quad \Rightarrow \quad \frac{a}{c} \ll \frac{a}{v} \quad \Rightarrow \quad \frac{v}{c} \ll 1$ slow-motion approximation

First term in a multipolar expansion (radiative zone $\lambda_{\rm GW} \ll R_0$):

$$\bar{h}_{\mu\nu} \simeq \frac{4G}{c^4}\frac{1}{R_0}\int T_{\mu\nu}(\mathbf{y}, t - \frac{R_0}{c})\, d^3y$$

Quadrupolar wave generation in linearized theory [continued]

For systems with significant self-gravity:

$$\bar{h}_{\mu\nu} \simeq \frac{4G}{c^4}\frac{1}{R_0}\int (T_{\mu\nu}+\tau_{\mu\nu})(\mathbf{y}, t-\frac{R_0}{c})\, d^3y$$

$$t^{\mu\nu} = T^{\mu\nu} + \tau^{\mu\nu} \quad \text{and} \quad \partial_\nu t^{\mu\nu} = 0$$

We disregard $\tau_{\mu\nu}$, impose $\partial_\nu T^{\mu\nu} = 0$ and show that

$\int T_{ij} dV$ **can be expressed** *only* **in terms of** T^{00}

Derivation of quadrupole formula

Eq. (1): $$\frac{\partial T_{0i}}{\partial x^i} - \frac{\partial T_{00}}{\partial x^0} = 0$$

Eq. (2): $$\frac{\partial T_{ji}}{\partial x^i} - \frac{\partial T_{j0}}{\partial x^0} = 0$$

multiplying Eq. (2) by x^k and integrating on all space

$$\int x^k \frac{\partial T_{ji}}{\partial x^i} dV = \int x^k \frac{\partial T_{j0}}{\partial x^0} dV = \frac{\partial}{\partial x^0} \int x^k T_{j0} dV$$

integrating by parts the LHS and assuming that the source decays sufficiently fast at ∞

$$-\int T_{ji}\, \delta_i^k\, dV = \frac{\partial}{\partial x^0} \int x^k T_{j0} dV$$

symmetrizing $\Rightarrow \int T_{kj}\, dV = -\frac{1}{2}\frac{\partial}{\partial x^0} \int (x_k T_{j0} + x_j T_{k0}) dV$

Derivation of quadrupole formula [continued]

Eq. (1): $$\frac{\partial T_{0i}}{\partial x^i} - \frac{\partial T_{00}}{\partial x^0} = 0$$

Eq. (2): $$\frac{\partial T_{ji}}{\partial x^i} - \frac{\partial T_{j0}}{\partial x^0} = 0$$

multiplying Eq. (1) by $x_k\, x_j$ and integrating on all space

$$\frac{\partial}{\partial x^0} \int T_{00}\, x_k\, x_j dV = \int \frac{\partial T_{0i}}{\partial x^i}\, x_k\, x_j\, dV$$

integrating by parts the RHS and assuming that the source decays sufficiently fast at ∞

$$\frac{\partial}{\partial x^0} \int T_{00}\, x_k\, x_j dV = -\int (x_k\, T_{j0} + x_j\, T_{k0}) dV$$

combining $\Rightarrow$ $$\int T_{kj} dV = \frac{1}{2c^2} \frac{\partial^2}{\partial t^2} \int T_{00}\, x_k\, x_j\, dV$$

Derivation of quadrupole formula [continued]

$$T^{00} = \mu\, c^2 \quad \Rightarrow \quad \bar{h}_{ij} = \frac{2G}{c^4}\,\frac{1}{R_0}\,\frac{\partial^2}{\partial t^2}\int \mu\, x_i\, x_j\, dV$$

Other components of $\bar{h}_{\mu\nu}$ are non-radiative (static) fields:

$$\bar{h}_{00} = \frac{4G}{c^2}\,\frac{1}{R_0}\,\underbrace{\int \mu dV}_{M} \qquad \bar{h}_{0k} = \frac{4G}{c^3}\,\frac{1}{R_0}\,\underbrace{\frac{\partial}{\partial t}\int \mu\, x_k\, dV}_{P}$$

In TT gauge: $$h_{ij}^{\rm TT} = \frac{2G}{c^4}\,\frac{1}{R_0}\,\mathcal{P}_i^k\,\mathcal{P}_j^l\,\ddot{Q}_{kl}$$

with $Q_{kl} = \int d^3x\rho\,\left(x_k\, x_l - \frac{1}{3}x^2\,\delta_{kl}\right) \qquad \mathcal{P}^{ik} = \delta^{ik} - n^i\, n^k$

$\bar{h}_{0k}$ component

Eq. (1): $$\frac{\partial T_{0i}}{\partial x^i} - \frac{\partial T_{00}}{\partial x^0} = 0$$

Eq. (2): $$\frac{\partial T_{ji}}{\partial x^i} - \frac{\partial T_{j0}}{\partial x^0} = 0$$

multiplying Eq. (1) by x_k and integrating on all space

$$\frac{\partial}{\partial x^0} \int T_{00}\, x_k\, dV = \int \frac{\partial T_{0i}}{\partial x^i}\, x_k\, dV$$

integrating by parts the RHS and assuming that the source decays sufficiently fast at ∞

$$\int T_{0k} dV = -\frac{\partial}{\partial x^0} \int T_{00}\, x_k dV$$

Total power radiated in GWs

Power radiated per unit solid angle in the direction $\mathbf{n}$:

$$\frac{dP}{d\Omega} = R_0^2\, n^i\, \tau^{i0} \qquad \textbf{with} \qquad \tau^{i0} = \frac{c^4}{32\pi G} < \partial_0 \bar{h}^\beta_\alpha\, \partial_i \bar{h}^\alpha_\beta >$$

- $$\frac{dP}{d\Omega} = \frac{G}{8\pi\, c^5} \left(\dddot{Q}_{ij}\, \epsilon^{ij}\right)^2$$ **for a given polarization**

$$\epsilon^{kk} = 0 \qquad \epsilon^{kl}\, n_k = 0 \qquad \epsilon^{kl}\, \epsilon_{kl} = 0$$

- $$\mathcal{L}_{\rm GW} \equiv P = \frac{G}{5\, c^5} \left(\dddot{Q}_{ij}\right)^2$$ **averaging over polarizations**

Useful relations

$$\overline{n_i\, n_j} = \frac{1}{3}\delta_{ij}$$

$$\overline{n_i\, n_j\, n_k\, n_l} = \frac{1}{5}(\delta_{ij}\,\delta_{kl} + \delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk})$$

$$\begin{aligned} 2\,\overline{\epsilon_{ij}\,\epsilon_{kl}} = \; & 2\,\frac{1}{4}\,\{n_i\, n_j\, n_k\, n_l + n_i\, n_j\,\delta_{kl} + n_k\, n_l\,\delta_{ij} - \\ & (n_i\, n_k\,\delta_{jl} + n_j\, n_k\,\delta_{il} + n_i\, n_l\,\delta_{jk} + n_j\, n_l\,\delta_{ik}) \\ & -\delta_{ij}\,\delta_{kl} + \delta_{ik}\,\delta_{jl} + \delta_{jk}\,\delta_{il}\} \end{aligned}$$

Comparison between GW and EM luminosity

$$\mathcal{L}_{\rm GW} = \frac{G}{5c^5}\,(\dddot{Q}_{ij})^2 \qquad Q \sim \epsilon\, M\, R^2$$

$R \to$ **typical source's dimension,** $M \to$ **source's mass,** $\epsilon \to$ **deviation from sphericity**

$\dddot{Q} \sim \omega^3\, \epsilon\, M\, R^3$ **with** $\omega \sim 1/\tau_{\rm source} \quad \Rightarrow \quad \mathcal{L}_{\rm GW} \sim \frac{G}{c^5}\,\epsilon^2\,\omega^6\,M^2\,R^4$

$\Rightarrow \quad \left[\frac{G}{c^5}\right]^{-1} = 3.6 \times 10^{50}$ **ergs/sec (huge!)**

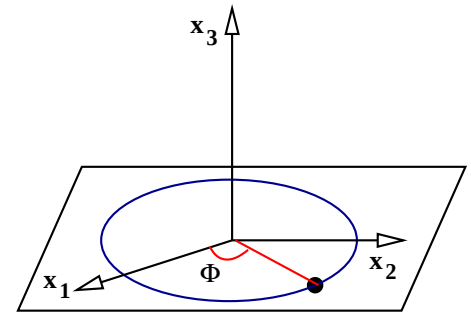
- **MTW evaluated that for a steel rod of** $M = 490$ **tons,** $R = 20$ **m and** $\omega \sim 28$ **rad/sec the luminosity** $\mathcal{L}_{\rm GW} \sim 10^{-22}$ **ergs/sec** $\sim 10^{-55}\mathcal{L}^{\rm EM}_{\rm sun}$ **(very tiny!)**
- **As Weber noticed in 1972, if we introduce** $R_S = 2GM/c^2$ **and** $\omega = (v/c)\,(c/R)$

$$\mathcal{L}_{\rm GW} = \frac{c^5}{G}\,\epsilon^2\,\left(\frac{v}{c}\right)^6\,\left(\frac{R_S}{R}\right) \quad \underbrace{\overset{\epsilon\sim 0.1}{\Rightarrow}}_{v\sim c,\, R\sim R_S} \quad \mathcal{L}_{\rm GW} \sim \frac{c^5}{G} \sim 10^{15}\,\mathcal{L}^{\rm EM}_{\rm sun}!$$

Electromagnetic astronomy versus gravitational-wave astronomy

EM astronomy	GW astronomy
– accelerating charges; time changing dipole	– accelerating masses; time changing quadr.
– incoherent superposition of emissions from electrons, atoms and molecules	– coherent superposition of radiation from bulk dynamics of dense source
– direct information about thermodynamic state	– direct information of system's dynamics
– wavelength small compared to source	– wavelength large compared to source
– absorbed, scattered, dispersed by matter	– very small interaction with matter
– frequency range: 10 MHz and up	– frequency range: 10 kHz and down

Binaries of black holes and/or neutron stars on circular orbit



Treating the two compact bodies as point particles with relative distance $\mathbf{x}$ and reduced mass μ:

$x_1 = r\,\cos\Phi$ and $x_2 = r\,\sin\Phi$

$$Q_{ij} = \mu\,(x_i\,x_j - \tfrac{1}{3}\delta_{ij}\,x^2)$$

$$Q_{11} = \mu r^2\,(\cos^2\Phi - 1/3), \qquad Q_{22} = \mu r^2\,(\sin^2\Phi - 1/3)$$

$$Q_{33} = -\mu r^2/3, \qquad Q_{12} = \mu r^2\,\sin\Phi\,\cos\Phi$$

$\dot{\Phi} = \sqrt{\frac{GM}{r^3}} = \omega$ [Newton law: $\omega^2\,r = \frac{GM}{r^2}$]

taking 3 time derivatives of $Q \Rightarrow \mathcal{L}_{\rm GW} = -\frac{dE}{dt} = \frac{32}{5}\frac{G^4}{c^5}\,\frac{\mu^2\,M^3}{r^5}$

Binary coalescence time

$$E = \frac{1}{2}\mu\, v^2 - \frac{G\mu\, M}{r} = -\frac{G\mu\, M}{2r} \quad \Rightarrow \quad r = -\frac{G\mu\, M}{2E}$$

$$\dot{r} = \frac{dr}{dE}\frac{dE}{dt} = -\frac{64}{5}\frac{G\mu\, M^2}{r^3} \quad \text{integrating} \quad \Rightarrow \quad r(t) = \left(r_0^4 - \frac{256}{5}\, G\mu\, M^2\, \Delta\tau_{\rm coal}\right)^{1/4}$$

If $\quad r(t_f) \ll r_0 \quad \Rightarrow \quad \Delta\tau_{\rm coal} = \frac{5}{256}\frac{r_0^4}{G\mu\, M^2}$

Examples:

- **LIGO/VIRGO/GEO/TAMA source:** $M = (10+10)M_\odot$ **at** $r_0 \sim 500$ **km,**

  $f_{\rm GW} \sim 40$**Hz,** $\quad T_0 \sim 0.05$**sec** $\quad \Rightarrow \quad \Delta\tau_{\rm coal} \sim 1\,{\rm sec}$

- **LISA source:** $M = (10^6 + 10^6)M_\odot$ **at** $r_0 \sim 200 \times 10^6\,{\rm km}$**,**

  $f_{\rm GW} \sim 4.5 \times 10^{-5}$ **Hz,** $\quad T_0 \sim 11$ **hours** $\quad \Rightarrow \quad \Delta\tau_{\rm coal} \sim 1\,{\rm year}$

GW radiated in non-precessing binaries in eccentric orbits

From Peters 64:

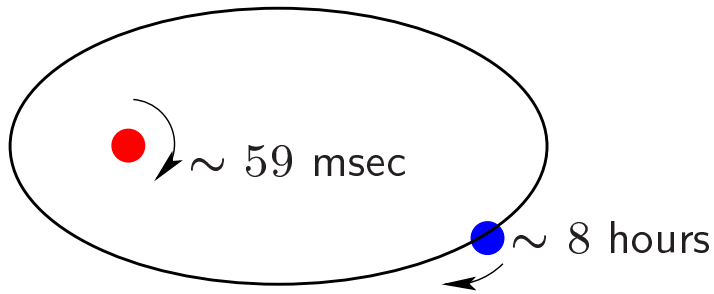
$$-\frac{dE}{dt} = \frac{32}{5}\frac{G^4}{c^5}\frac{m_1^2\, m_2^2\, M}{a^5\,(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right) \qquad a = \frac{r_1+r_2}{2}$$

$e \quad \rightarrow \quad$ eccentricity $\qquad a \quad \rightarrow \quad$ semimajor axis

Indirect observation of gravitational wa

Neutron Binary System: PSR 1913 +16 - Timing Pulsars

Hulse & Taylor discovery (1974)



Separated by $\sim 10^6$ Km, $m_1 = 1.4 M_\odot$, $m_2 = 1.36 M_\odot$, $e = 0.617$

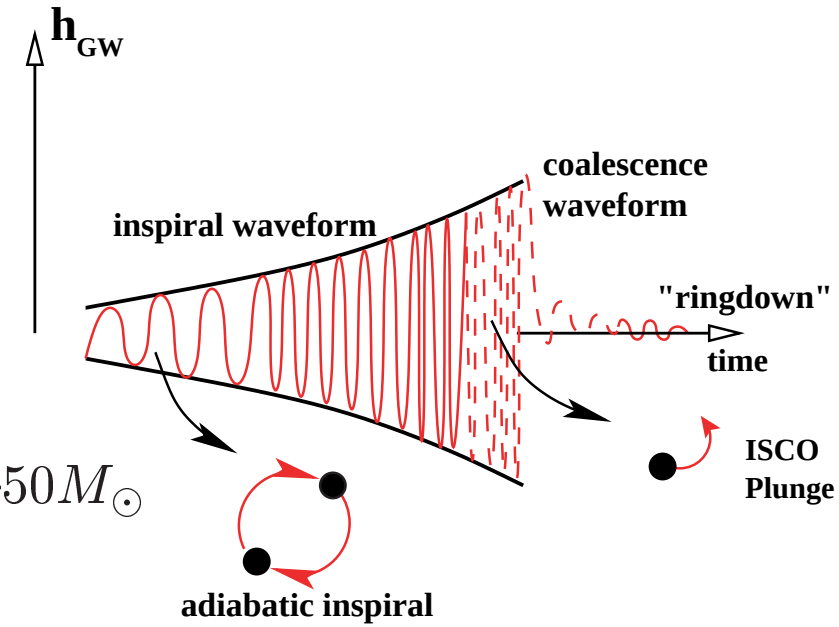
- **Prediction from GR: rate of change of orbital period**
- **Emission of gravitational waves:**
  - **due to loss of orbital energy**
  - **orbital decay in very good agreement with GR**

Gravitational waves from compact binaries

- **GW signal: "chirp"** [duration $\sim$ seconds to years] ($f_{\rm GW} \sim 10^{-4}\,{\rm Hz}$–$1{\rm kHz}$)

– **NS/NS, NS/BH and BH/BH**

– **MACHO binaries ($m < 1 M_\odot$)** [MACHOs in galaxy halos $\lesssim 3$–5%]



GW frequency at end-of-inspiral (ISCO):

$$f_{\rm GW} \sim 4400 \frac{M_\odot}{M}\ {\rm Hz}$$

- **LIGO/VIRGO/GEO/TAMA:** $M = 1$–$50 M_\odot$
- **LISA:** $M = 10^2$–$10^7 M_\odot$

Inspiral signals are "chirps"

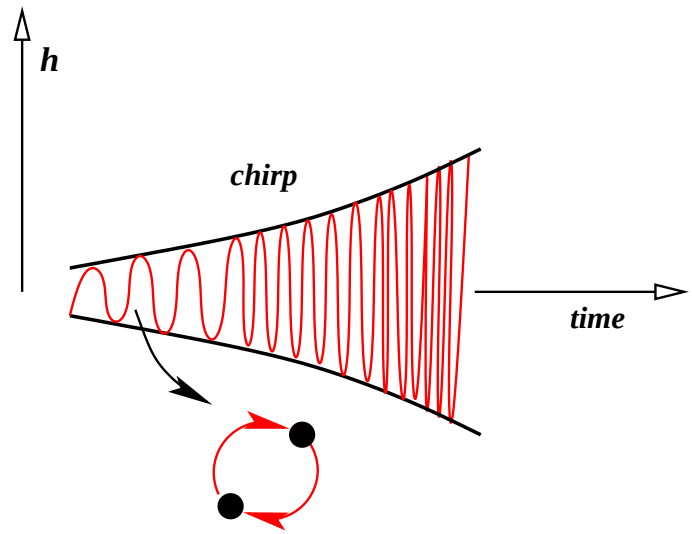
- Mass-quadrupole approximation: $h_{ij}^{\rm TT} = \frac{2G}{R_0 c^4} \mathcal{P}_i^k \mathcal{P}_j^l \ddot{Q}_{kl}$

$$Q_{kl} = \int d^3x \, \rho \, (x_k \, x_l - r^2 \, \delta_{kl})$$

$$h_{\rm GW} \propto \omega^{2/3} \cos 2\Phi$$

for quasi-circular orbits: $\omega^2 = \dot{\Phi}^2 = \frac{GM}{r^3}$

$$\dot{\omega} \propto \omega^{11/3}$$

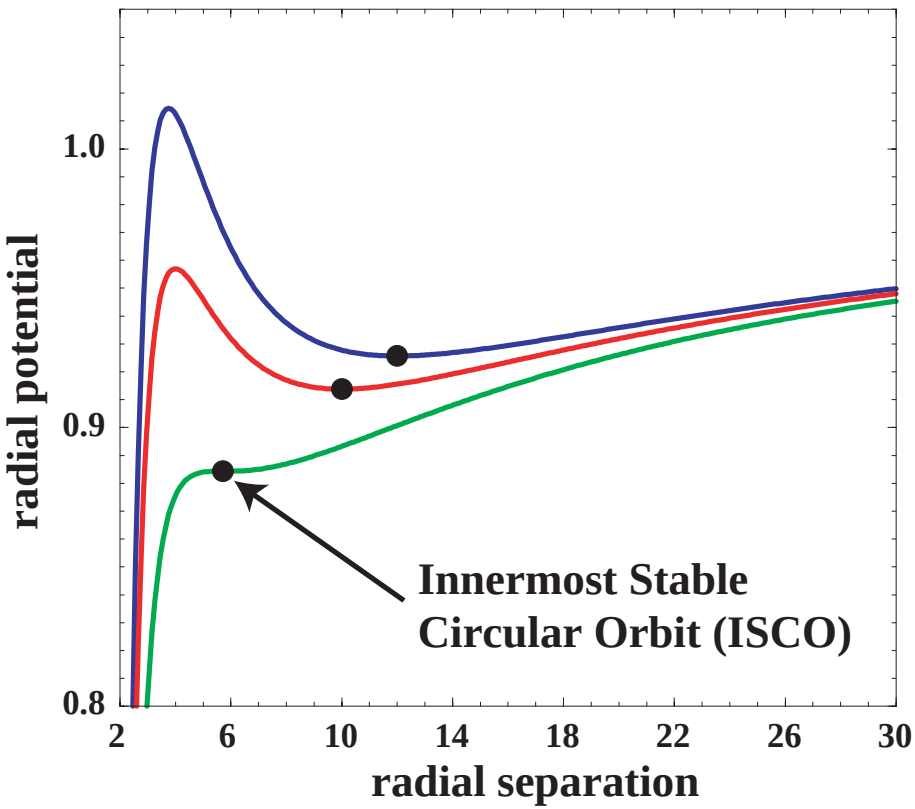


Chirp: The signal continuously changes its frequency and the power emitted at any frequency is very small!

$h \sim \frac{M^{5/3} f_{\rm GW}^{2/3}}{R_0}$ for $f_{\rm GW} \sim 100$ Hz, $M = 20 M_\odot$

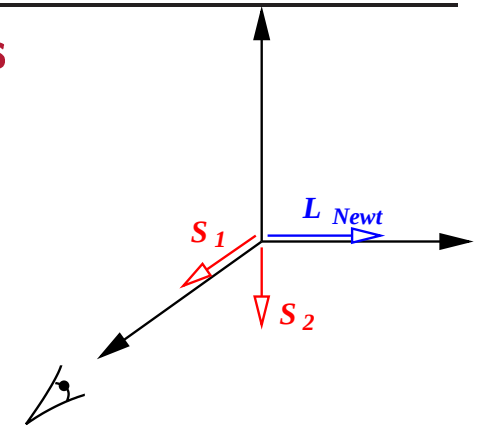
R_0 at 100 Mpc $\Rightarrow h \sim 10^{-21}$

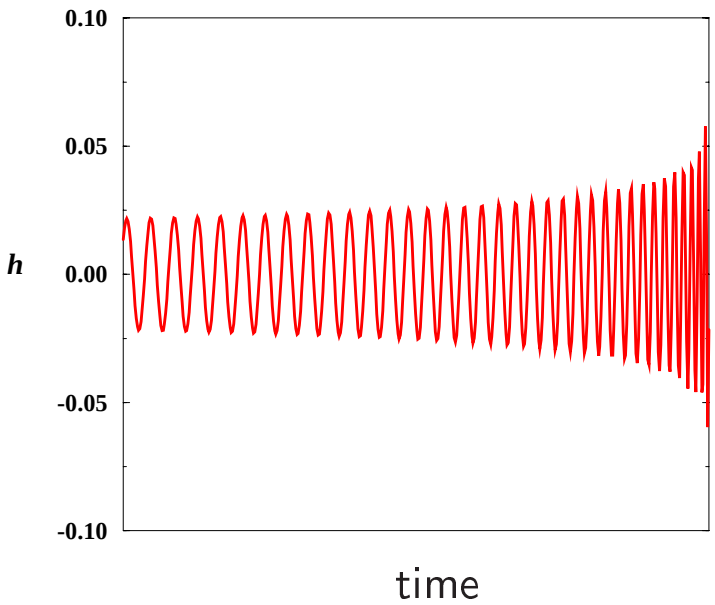
Radial potential in black-hole spacetime

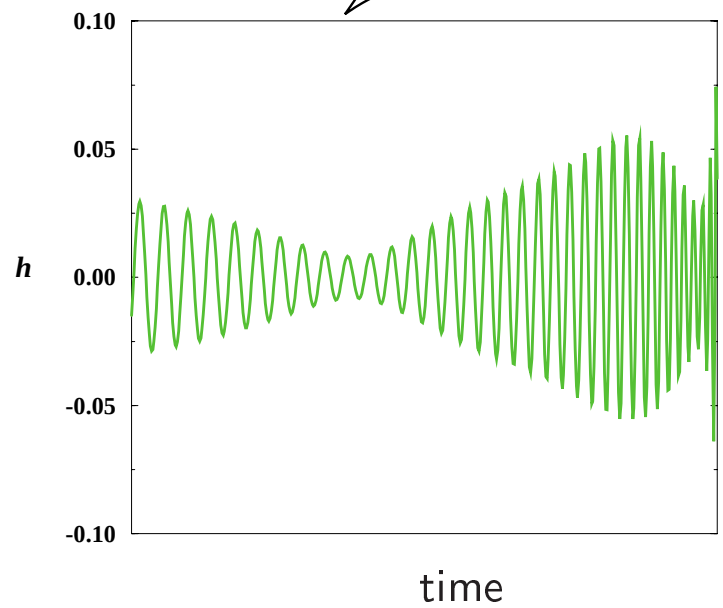


Waveforms including spin effects

Maximal spins $M = 10M_\odot + 10M_\odot$







Gravitational waves from stellar collapse

- **GW signal: "bursts"** [$\sim$ few mseconds] **or (quasi) "periodic"** ($f_{\rm GW} \sim 1\,{\rm kHz}$–$10{\rm kHz}$)

Supernovae:

- Non-axisymmetric core collapse
- Material in the stellar core may form a rapidly rotating bar-like structure
- Collapse material may fragment into clumps which orbit as the collapse proceeds
- Pulsation modes of new-born NS; ring-down of new-born BH

Dynamics of star very complicated

- GW amplitude and frequency estimated using mass- and current-quadrupole moments
- Numerical simulations

Correlations with neutrino flux and/or EM counterparts

Event rates in our galaxy and its companions $\lesssim 30$ yrs

Gravitational-wave strain from non-axisymmetric collapse

$$h_{\rm GW} \simeq 2 \times 10^{-17} \sqrt{\eta_{\rm eff}} \left(\frac{1\,{\rm msec}}{\tau}\right)^{1/2} \left(\frac{M}{M_\odot}\right)^{1/2} \left(\frac{10\,{\rm kpc}}{R_0}\right) \left(\frac{1\,{\rm kHz}}{f_{\rm GW}}\right)$$

$\tau \rightarrow$ duration of emission

efficiency $\eta_{\rm eff} = \frac{\Delta E}{M\,c^2} \sim 10^{-10}$–$10^{-7}$

Gravitational waves from spinning neutron stars: pulsars

- **GW signal: (quasi) "periodic"** ($f_{\rm GW} \sim 10\,{\rm Hz}$–$1\,{\rm kHz}$)

Pulsars: non-zero ellipticity (or oblateness)

$$h_{\rm GW} \simeq 7.7 \times 10^{-26} \left(\frac{\epsilon}{10^{-6}}\right) \left(\frac{I_{33}}{10^{45}\,{\rm g\,cm^2}}\right) \left(\frac{10\,{\rm kpc}}{R_0}\right) \left(\frac{f_{\rm GW}}{1\,{\rm kHz}}\right)^2$$

$\epsilon = \frac{I_{11} - I_{22}}{I_{33}} \rightarrow$ ellipticity

– **The crust contributes only 10% of total moment of inertia $\Rightarrow \epsilon_C$ is low**

– **Magnetic fields could induce stresses and generate $\epsilon_M \neq 0$**

Expected ellipticity rather low $\leq 10^{-7}$

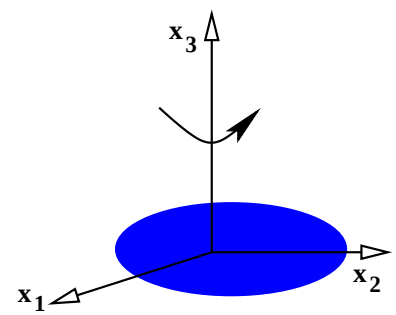
- **search for known spinning neutron stars: Vela, Crab, ...**
- **all sky search**

Gravitational waves from pulsars and planets !

- Body rotating rigidly around the x_3-axis with angular frequency Ω

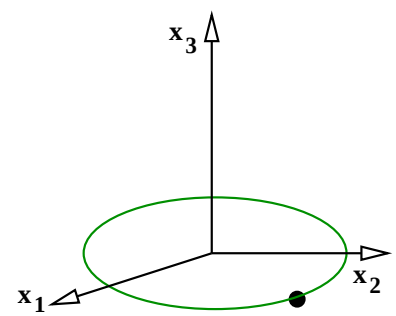
$\{x'_1, x'_2, x'_3\}$ coordinate system fixed to the body

$$x_1 = x'_1 \cos\Omega t - x'_2 \sin\Omega t$$
$$x_2 = x'_1 \sin\Omega t + x'_2 \cos\Omega t$$



$$Q_{ij} = \int \rho\, x_i\, x_j\, d^3x \quad \text{and} \quad I_{ij} = \int \rho\, x'_i\, x'_j\, d^3x'$$

$$Q_{11} = \tfrac{1}{2}\left(I_{11} + I_{22}\right) + \tfrac{1}{2}\left(I_{11} - I_{22}\right) \cos 2\Omega t$$
$$Q_{22} = \tfrac{1}{2}\left(I_{11} + I_{22}\right) + \tfrac{1}{2}\left(I_{22} - I_{11}\right) \cos 2\Omega t$$
$$Q_{33} = I_{33}, Q_{13} = Q_{23} = 0$$



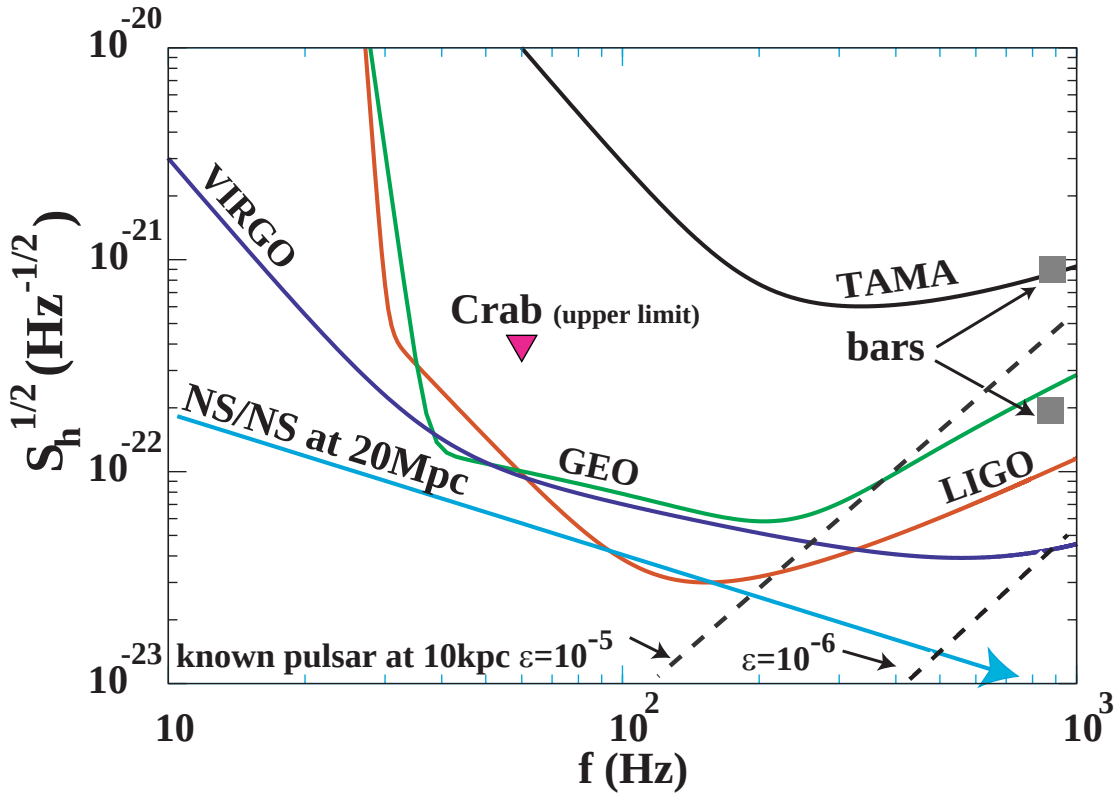
$$h \sim \frac{2\Omega^2}{R_0}\left(I_{11} - I_{22}\right) \cos 2\Omega t \qquad \epsilon \equiv (I_{11} - I_{22})/I_{33}$$

- Body fixed at distance r $(I_{11} = mr^2, \epsilon = 0) \Rightarrow$

$$P = \frac{32G}{5c^5}\,\Omega^6\, m^2\, r^4 \sim 50 \text{ ergs/sec for Jupiter!}$$

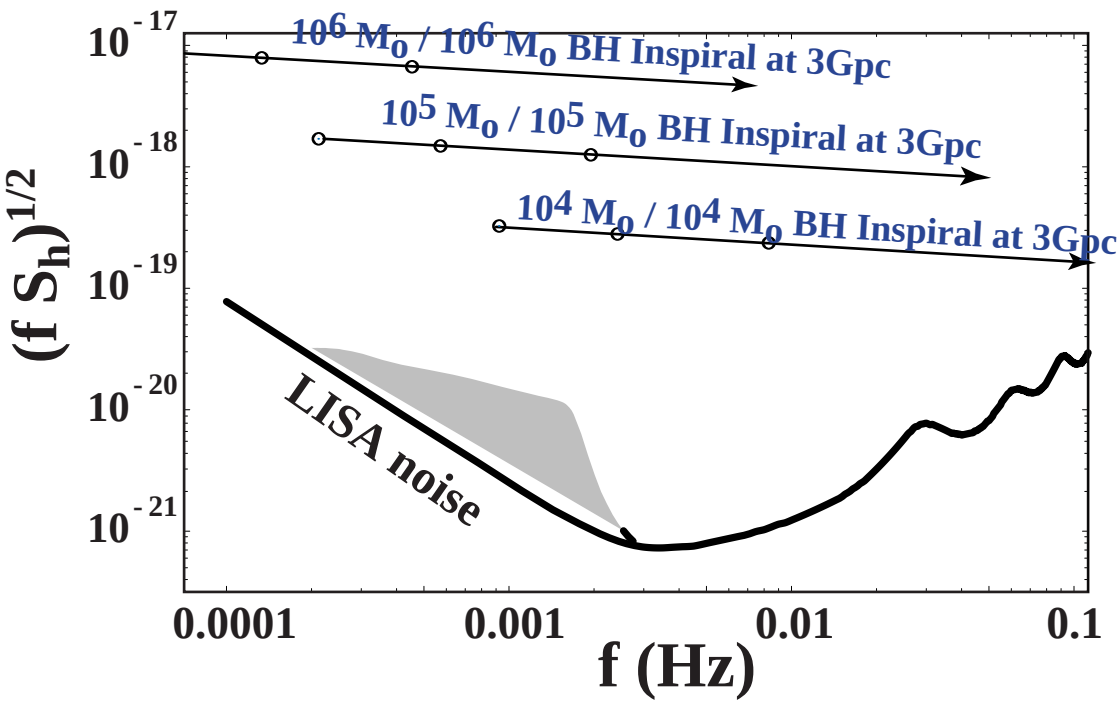
Sources for first generation ground-based detectors

$$\sqrt{S_h \, \Delta f} = h = \frac{\delta \xi}{L}$$
$$\Delta f \sim f$$



Supermassive black-hole binaries and LISA

$$\sqrt{S_h \, \Delta f} = h = \frac{\delta \xi}{L}$$
$$\Delta f \sim f$$



GWs in curved space-time

$$S = \frac{1}{16\pi\, G_N} \int d^4x \sqrt{|g|}\, \mathcal{R} + S_{\rm matter}$$

- **Isotropic and spatially homogenous FLRW background**

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -dt^2 + a^2(t)\, d\vec{x}^2 = a^2(\eta)\, (-d\eta^2 + d\vec{x}^2)$$

- **Metric perturbations** $(\delta g_{\mu\nu} = h_{\mu\nu})$:

$$h_k''(\eta) + \frac{2a'}{a}\, h_k'(\eta) + k^2\, h_k(\eta) = 0$$

Introducing the "canonical field" $\psi_k(\eta) = a\, h_k(\eta)$:

$$\psi_k'' + \left[k^2 - U(\eta)\right]\, \psi_k = 0 \qquad U(\eta) = \frac{a''}{a}$$

Semiclassical point of view

Introducing the "canonical field" $\psi_k(\eta) = a\, h_k(\eta)$:

$$\psi_k'' + \left[k^2 - U(\eta)\right]\, \psi_k = 0 \qquad U(\eta) = \frac{a''}{a}$$

"deSitter-like" inflationary era: $a = -1/(\eta\, H_{\rm dS})$ $\left[|U(\eta)| \sim 1/\eta^2,\ (a\, H_{\rm dS}) \sim 1/\eta\right]$

- If $k^2 \gg |U(\eta)|$

  $\left[k\eta \gg 1\,,\, k/a \gg H_{\rm dS}\,,\, \lambda_{\rm phys} \ll H_{\rm dS}^{-1} \rightarrow \right.$ the mode is inside the Hubble radius$\left.\right]$

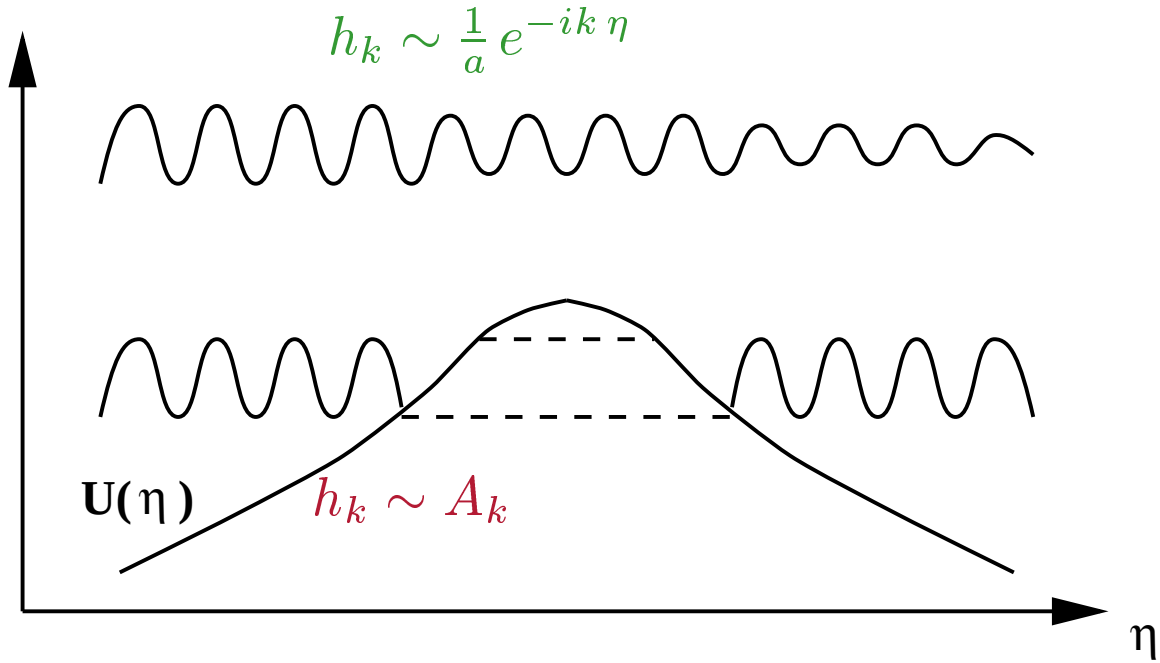
  $$\psi_k \sim e^{\pm ik\,\eta} \Rightarrow \quad h_k \sim \frac{1}{a}\, e^{-ik\,\eta}$$

- If $k^2 \ll |U(\eta)|$:

  $\left[k\eta \ll 1\,,\, k/a \ll H_{\rm dS}\,,\, \lambda_{\rm phys} \gg H_{\rm dS}^{-1} \rightarrow \right.$ the mode is outside the Hubble radius$\left.\right]$

  $$\psi_k \sim a\, \left[A_k + B_k \int \frac{d\eta}{a^2(\eta)}\right] \quad \Rightarrow \quad h_k \sim A_k + B_k \int \frac{d\eta}{a^2(\eta)}$$

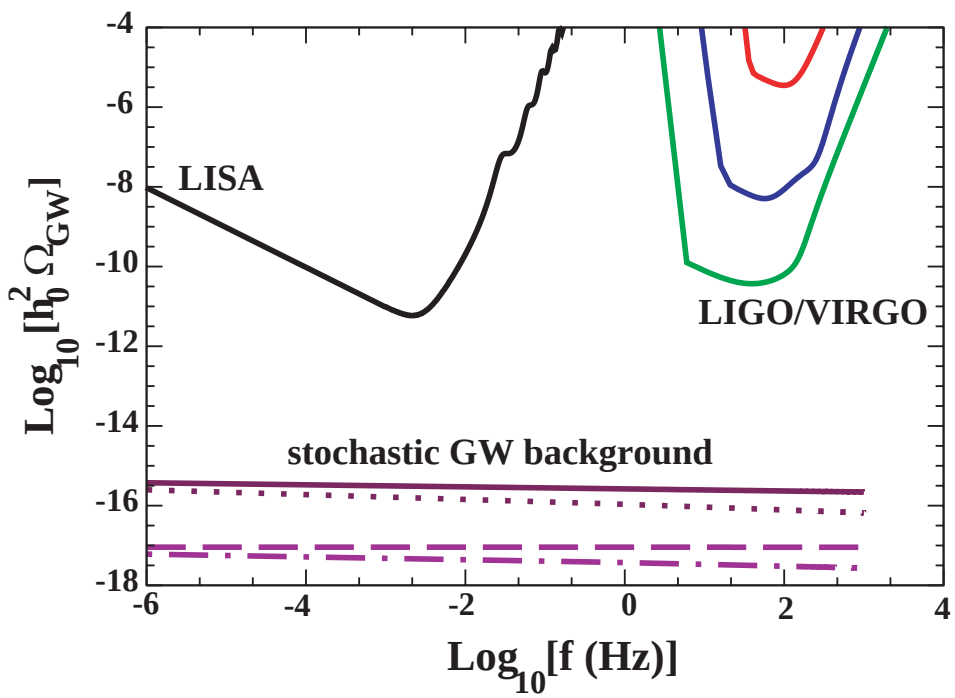
Amplification of quantum-vacuum fluctuations: semiclassical point of view



Example: Slow-roll inflation

- **Too low to be detected by first generations GW interferometers**

$$h_0^2\Omega_{\rm GW} = \frac{1}{\rho_c}\frac{d\rho_{\rm GW}}{d\log f}$$



Example: Slow-roll inflation

- **Might be detected from polarization of CMB**