The Universe is filled with a bath of thermal radiation.

COBE map of the CMB temperature

On large scales, the CMB temperature is nearly isotropic around us (the same in all directions): snapshot of the young Universe, \( t \sim 380,000 \) years.

Temperature fluctuations

\[ \frac{\delta T}{T} \sim 10^{-5} \sim \delta \Phi_{\text{grav}} \]

\[ T = 2.728 \text{ degrees above absolute zero} \]
The Cosmological Principle

• We are not privileged observers at a special place in the Universe.

• At any instant of time, the Universe should appear **ISOTROPIC** (averaged over large scales) to all Fundamental Observers (those who define local standard of rest).

• A Universe that appears isotropic to all FO’s is **HOMOGENEOUS** the same at every location (averaged over large scales). **Space-like hypersurfaces of constant density, temperature, etc.**
The only mode that preserves homogeneity and isotropy is overall expansion or contraction:

Cosmic scale factor $a(t)$

Model completely specified by $a(t)$ and sign of spatial curvature
On average, galaxies are at rest in these expanding (comoving) coordinates, and they are not expanding--they are gravitationally bound.

Wavelength of radiation scales with scale factor:

$$\lambda \sim a(t)$$

Redshift of light:

$$1 + z = \frac{\lambda(t_2)}{\lambda(t_1)} = \frac{a(t_2)}{a(t_1)}$$

indicates relative size of Universe directly
Distance between galaxies:
\[ d(t) = a(t)r \]

where \( r \) = fixed comoving distance

Recession speed:
\[
\begin{align*}
    v &= \frac{dd(t)}{dt} = r \frac{da}{dt} \\
    &= \frac{d}{a} \frac{da}{dt} \equiv dH(t) \\
    v &\approx cz \approx dH_0 \text{ for `small' } d
\end{align*}
\]

\[ H_0 \equiv \left( \frac{\dot{a}}{a} \right)_0 \text{ and 0 denotes today} \]

Hubble’s Law (1929)
Hubble Parameter

- \( H_0 = 72 \text{ km/sec/Mpc} = 100 \ h \text{ km/sec/Mpc}, \quad h=0.7 \)

- \( t_H = \frac{1}{H_0} = 9.8h^{-1} \text{ Gyr} \) \quad Hubble time

- \( d_H = \frac{c}{H_0} = 3000 \ h^{-1} \text{ Mpc} \) \quad Hubble distance

- Distances: \( d = \frac{v}{H_0} = \frac{cz}{H_0} = d_H z \)

- Density: \( \rho_{\text{crit}} = \frac{3H_0^2}{8\pi G} = 1.88h^2 \times 10^{-29} \text{ gm/cm}^3 \)
Expansion Kinematics

- Taylor expand about present epoch:

$$a(t) = a(t_0) + \dot{a}(t_0)(t - t_0) + \frac{1}{2} \ddot{a}(t_0)(t - t_0)^2 + ...$$

Which implies to 2nd order in $t - t_0$:

$$\frac{a(t)}{a_0} = 1 + \left( \frac{\dot{a}}{a} \right)_0 (t - t_0) + \frac{1}{2} \left( \frac{\ddot{a}}{a} \right)_0 (t - t_0)^2 = 1 + H_0(t - t_0) - \frac{q_0 H_0^2}{2} (t - t_0)^2$$

Where $H(t) = \dot{a}/a(t)$, $H_0 = (\dot{a}/a)|_{t=t_0}$ and $q_0 \equiv -(a\ddot{a}/\dot{a}^2)_0$

Differentiating with respect to $t$ and keeping terms linear in $t - t_0$,

$$H(t) = \frac{\dot{a}(t)}{a(t)} = \frac{\dot{a}(t)}{a_0} \frac{a_0}{a(t)} = (H_0 - q_0 H_0H_0^2 + q_0 H_0^3 t_0) (1 - H_0(t - t_0))$$

$$= H_0 [1 - (1 + q_0) H_0(t - t_0)]$$

The redshift is given generally by

$$1 + z = \frac{a_0}{a(t)} ,$$

so that to this order of approximation we have

$$z = -H_0(t - t_0) + \mathcal{O}(t - t_0)^2 ,$$

and we therefore find to this order,

$$H(z) = H_0 [1 + (1 + q_0) z] .$$

Recent expansion history completely determined by $H_0$ and $q_0$
Cosmological Dynamics

How does the scale factor of the Universe evolve?

Consider a homogenous ball of matter:

Kinetic Energy \( \frac{mv^2}{2} \)

Gravitational Energy \( -\frac{GMm}{d} \)

Conservation of Energy:

\[
E = \frac{mv^2}{2} - \frac{GMm}{d}
\]

Birkhoff’s theorem

Substitute \( v = Hd \) and \( M = \frac{(4\pi/3)\rho d^3}{6} \) to find

\[
\frac{2E}{md^2} = -\frac{K}{a^2(t)} = H^2(t) - \left(\frac{8\pi}{3}\right)G\rho(t)
\]

1st order Friedmann equation

\( K \) interpreted as spatial curvature in General Relativity
Local Conservation of Energy-Momentum

First law of thermodynamics:

\[ dE = -p \, dV \]

Energy:

\[ E = \rho V \sim \rho a^3 \]

First Law becomes:

\[ \frac{d}{dt}(\rho a^3) = -p \frac{d}{dt}(a^3) \]

\[ a^3 \dot{\rho} + 3\rho a^2 \dot{a} = -3pa^2 \dot{a} \quad \Rightarrow \]

\[ \frac{d\rho_i}{dt} + 3H(t)(p_i + \rho_i) = 0 \]
Equation of State parameter \( w \) determines Cosmic Evolution

\[
w_i(z) = \frac{p_i}{\rho_i}
\]

\[
\dot{\rho}_i + 3H\rho_i(1 + w_i) = 0
\]

Conservation of Energy-Momentum

\( w_r = 1/3, \rho_r \sim a^{-4} \)

\( \rho_m \sim a^{-3} \)

\( \rho_{DE} \sim a^{-3(1+w_{DE})}, w_{DE} < -1/3 \)
2\textsuperscript{nd} Order Friedmann Equation

First order Friedmann equation:
\[ \ddot{a}^2 = \frac{8\pi G}{3} \rho a^2 - k \]

Differentiate:
\[ 2\dot{a}\ddot{a} = \frac{8\pi G}{3} (a^2 \dot{\rho} + 2a\dot{\rho}a\dot{a}) \]
\[ \ddot{a} = \frac{4\pi G}{3} \left[ \dot{\rho} \left( \frac{a}{\dot{a}} \right) + 2\rho \right] \]

Now use conservation of energy - momentum:
\[ \frac{d\rho_i}{dt} + 3H(t)(p_i + \rho_i) = 0 \]

2\textsuperscript{nd} order Friedmann equation:
\[ \ddot{a} = \frac{4\pi G}{3} \left[ -3(p + \rho) + 2\rho \right] = -\frac{4\pi G}{3} \left[ \rho + 3p \right] \]
Cosmological Dynamics

\[ H^2(t) = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \sum_i \rho_i(t) - \frac{k}{a^2(t)} \]

\[ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \sum_i \left( \rho_i + \frac{3p_i}{c^2} \right) \]

Equation of state parameter: \( w_i = \frac{p_i}{\rho_i c^2} \)

Non-relativistic matter: \( p_m \sim \rho_m v^2 \), \( w_m \approx 0 \)

Relativistic particles: \( p_r = \rho_r c^2 / 3 \), \( w_r = 1/3 \)

Dark Energy: component with negative pressure: \( w_{DE} < -1/3 \)

Spatial curvature: \( k = 0, +1, -1 \)

Friedmann Equations valid in GR
The 2nd order Friedmann equation for a single component Universe gives
\[
\left( \frac{\ddot{a}}{a} \right)_0 = -\frac{4\pi G}{3} (\rho_0 + 3p_0) . \tag{18}
\]
From the first order Friedmann equation, the density parameter is given by
\[
\Omega_0 = \frac{\rho_0}{\rho_{\text{crit}}} = \frac{\rho_0}{3H_0^2/8\pi G} = \frac{8\pi G \rho_0}{3H_0^2} , \tag{19}
\]
so that
\[
H_0^2 = \frac{8\pi G \rho_0}{3 \Omega_0} . \tag{20}
\]
Combining Eqns. 18 and 20 gives the deceleration parameter,
\[
q_0 = - \left( \frac{a\ddot{a}}{a^2} \right)_0 = - \frac{\ddot{a}_0}{H_0^2a_0} = \frac{4\pi G}{3} (\rho_0 + 3p_0) \frac{3\Omega_0}{8\pi G \rho_0} \frac{3\Omega_0}{8\pi G \rho_0} = \frac{\Omega_0}{2} \left( 1 + \frac{3p_0}{\rho_0} \right) . \tag{21}
\]
For a multi-component Universe, this generalizes to
\[
q_0 = \frac{1}{2} \sum_i \Omega_i (1 + 3w_i) , \tag{22}
\]
where the equation of state parameter \( w_i = p_i/\rho_i \). For non-relativistic matter plus dark energy, this becomes
\[
q_0 = \frac{\Omega_m}{2} + \frac{\Omega_{DE}}{2} (1 + 3w) . \tag{23}
\]
Einstein-de Sitter Universe: a special case

Non-relativistic matter: \( w_m = 0, \rho_m \sim a^{-3} \)

Spatially flat: \( k = 0 \Rightarrow \Omega_m = 1 \)

Friedmann:

\[
\left( \frac{\dot{a}}{a} \right)^2 \sim \frac{1}{a^3} \Rightarrow a^{1/2} \, da \sim \, dt \Rightarrow a \sim t^{2/3} \\
H = \frac{2}{3t} \Rightarrow t_0 = \frac{2}{3H_0} = 9.3 \text{ Gyr for } H_0 = 70 \\
q_0 = 1/2
\]
Cosmological Observables

Friedmann-Robertson-Walker Metric:

\[ ds^2 = c^2 dt^2 - a^2(t) \left[ d\chi^2 + S_k^2(\chi) \left\{ d\theta^2 + \sin^2 \theta \, d\phi^2 \right\} \right] \]

\[ = c^2 dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 \left\{ d\theta^2 + \sin^2 \theta \, d\phi^2 \right\} \right] \]

where

\[ r = S_k(\chi) = \sinh(\chi), \chi, \sin(\chi) \text{ for } k = -1,0,1 \]

Comoving distance: (radial photon path)

\[ cdt = a \, d\chi \quad \Rightarrow \quad \chi = \int \frac{cdt}{a} = \int \frac{cdt}{ada} = c \int \frac{da}{a^2 H(a)} \]

\[ = c \int \frac{dz}{H(z)} \]
Age of the Universe

cdt = ad \chi

\[ t = \int ad \chi = \int \frac{da}{aH(a)} = \int \frac{dz}{(1 + z)H(z)} \]

\[ t_0 = \frac{1}{H_0} \int_0^\infty \frac{dz}{(1 + z)E(z)} \]

where \( E(z) = \frac{H(z)}{H_0} \)
Exercise 1:

\[ E^2(z) = \frac{H^2(z)}{H_0^2} = \Omega_m (1 + z)^3 + \Omega_{DE} \exp\left[3 \int (1 + w(z)) d\ln(1 + z)\right] + (1 - \Omega_m - \Omega_{DE})(1 + z)^2 \]

A. For \( w = -1 \) (cosmological constant \( \Lambda \)) and \( k = 0 \):

\[ E^2(a) = \frac{H^2(a)}{H_0^2} = \Omega_m a^{-3} + \Omega_{\Lambda} \]

Derive an analytic expression for \( H_0 t_0 \) in terms of \( \Omega_m \)

Plot \( H_0 t_0 \) vs. \( \Omega_m \)

B. Do the same, but for \( \Omega_{\Lambda} = 0, \Omega_k \neq 0 \)

C. Suppose \( H_0 = 70 \) km/sec/Mpc and \( t_0 = 13.7 \) Gyr.

Determine \( \Omega_m \) in the 2 cases above.

D. Repeat part C but with \( H_0 = 72 \).
Age of the Universe

\[ \frac{\Omega_M}{(H_0/72)^2} \times (\text{Gyr}) \]

\[ \Omega_M = 1 - \Omega_{DE} \text{ (flat)} \]

Globular clusters

\( w = -0.5 \)

WMAP
Distance and $q_0$

\[ H(z) = H_0 \left[ 1 + (1 + q_0)z \right]. \quad (11) \]

The coordinate distance is

\[ a_0 \chi = a_0 \int \frac{dt}{a(t)} = a_0 \int \frac{dt}{da} \frac{da}{a} = a_0 \int \frac{da}{H(a)a^2}. \quad (12) \]

Using Eqn. 9, this can be written as

\[ a_0 \chi(z) = \int \frac{dz}{H(z)}. \quad (13) \]

Using Eqn. 11, this becomes

\[ a_0 \chi(z) = \int \frac{dz}{H_0[1 + (1 + q_0)z]} \approx \frac{1}{H_0} \int dz [1 - (1 + q_0)z] = \frac{1}{H_0} \left[ z - (1 + q_0) \frac{z^2}{2} \right]. \quad (14) \]

The radial distance $r = \sin \chi, \chi, \sinh \chi$ for $k = +1, 0, -1$. For small distances, $\chi \ll 1$, this means $r = \chi \pm \mathcal{O}(\chi^3)$. Since, from Eqn. 14, $\chi \propto z + \mathcal{O}(z^2)$, the expression for $a_0 r(z)$ to $\mathcal{O}(z^2)$ is identical to the expression for $a_0 \chi(z)$ to the same order, i.e., Eqn. 14.
Distance and $q_0$

Recall

\[
q_0 = \frac{\Omega_m}{2} + \frac{\Omega_{DE}}{2} (1 + 3w)
\]

For a flat Universe, $\Omega_{DE} = 1 - \Omega_m$:

\[
q_0 = \frac{\Omega_m}{2} + \frac{(1 - \Omega_m)}{2} (1 + 3w) = \frac{1}{2} + \frac{3w}{2} (1 - \Omega_m),
\]

(25)

\[
H_0 r \approx z - \frac{3}{4} z^2 (1 + w[1 - \Omega_m]) + O(z^3)
\]
Coordinate Distance

\[ H_0r(z) \approx z - \frac{3}{4} z^2 (1 + w [1 - \Omega_m]) + O(z^3) \]
Angular Diameter Distance

• Observer at $r = 0$, $t_0$ sees source of proper diameter $D$ at coordinate distance $r = r_1$ which emitted light at $t = t_1$:

\[
\theta = \frac{D}{a_1 r_1}
\]

• From FRW metric, proper distance across the source is $D = a(t_1) r_1 \theta$ so the angular diameter of the source is

\[
\theta = \frac{D}{a_1 r_1}
\]

• In Euclidean geometry, $d = \frac{D}{\theta}$ so we define the Angular Diameter Distance:

\[
d_A \equiv \frac{D}{\theta} = a_1 r_1 = a_1 S_k (\chi_1) = \frac{r_1 a_0}{1 + z_1}
\]
Luminosity Distance

- Source $S$ at origin emits light at time $t_1$ into solid angle $d\Omega$, received by observer $O$ at coordinate distance $r$ at time $t_0$, with detector of area $A$:

  Proper area of detector given by the metric:
  
  $$A = a_0 r d\theta a_0 r \sin \theta d\phi = a_0^2 r^2 d\Omega$$

  Unit area detector at $O$ subtends solid angle $d\Omega = 1/a_0^2 r^2$ at $S$.

  Power emitted into $d\Omega$ is $dP = L d\Omega / 4\pi$

  Energy flux received by $O$ per unit area is
  
  $$f = \frac{L d\Omega}{4\pi} = \frac{L}{4\pi a_0^2 r^2}$$
Include Expansion

- Expansion reduces received flux due to 2 effects:
  
  1. Photon energy redshifts: \( E_\gamma(t_0) = \frac{E_\gamma(t_1)}{1 + z} \)
  
  2. Photons emitted at time intervals \( \delta t_1 \) arrive at time intervals \( \delta t_0 \):

\[
\int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_{t_1 + \delta t_1}^{t_0 + \delta t_0} \frac{dt}{a(t)}
\]

for source at const. \( \chi \)

\[
\int_{t_1}^{t_1 + \delta t_1} \frac{dt}{a(t)} + \int_{t_1 + \delta t_1}^{t_0} \frac{dt}{a(t)} = \int_{t_1 + \delta t_1}^{t_0} \frac{dt}{a(t)} + \int_{t_0}^{t_0 + \delta t_0} \frac{dt}{a(t)}
\]

\[
\frac{\delta t_1}{a(t_1)} = \frac{\delta t_0}{a(t_0)} \Rightarrow \frac{\delta t_0}{\delta t_1} = \frac{a(t_0)}{a(t_1)} = 1 + z
\]

\[
f = \frac{L \, d\Omega}{4\pi} = \frac{L}{4\pi a_0^2 r^2 (1 + z)^2} \equiv \frac{L}{4\pi d_L^2} \Rightarrow d_L = a_0 r (1 + z) = (1 + z)^2 d_A
\]

Convention: choose \( a_0 = 1 \) Luminosity Distance
Worked Example I

For $w = -1$ (cosmological constant $\Lambda$):

$$E^2(z) = \frac{H^2(z)}{H_0^2} = \Omega_m (1 + z)^3 + \Omega_{DE} \exp\left[3 \int (1 + w(z))d\ln(1 + z)\right] + (1 - \Omega_m - \Omega_{DE})(1 + z)^2$$

Luminosity distance:

$$d_L(z; \Omega_m, \Omega_\Lambda) = r(1 + z) = c(1 + z)S_k \left( \int \frac{da}{H_0 a^2 E(a)} \right)$$

$$= c(1 + z)S_k \left( \int \frac{da}{H_0 a^2 [\Omega_m a^{-3} + \Omega_\Lambda + (1 - \Omega_m - \Omega_\Lambda) a^{-2}]^{1/2}} \right)$$
Worked Example II

For a flat Universe \((k=0)\) and constant Dark Energy equation of state \(w\):

\[
E^2(z) = \frac{H^2(z)}{H_0^2} = \Omega_m (1 + z)^3 + \Omega_{DE} \exp \left[ 3 \int (1 + w(z)) d\ln(1 + z) \right] + (1 - \Omega_m - \Omega_{DE})(1 + z)^2
\]

Luminosity distance:

\[
E^2(z) = \frac{H^2(z)}{H_0^2} = (1 - \Omega_{DE})(1 + z)^3 + \Omega_{DE} (1 + z)^{3(1+w)}
\]

\[
d_L(z; \Omega_{DE}, w) = r(1 + z) = \chi(1 + z) = \frac{c(1 + z)}{H_0} \int \frac{da}{a^2 E(a)}
\]

\[
= \frac{c(1 + z)}{H_0} \int \frac{1 + \Omega_{DE}[(1 + z)^{3w} - 1]^{-1/2}}{(1 + z)^{3/2}} dz
\]

Note: \(H_0 d_L\) is independent of \(H_0\)
Dark Energy Equation of State

Curves of constant $d_L$ at fixed $z$

Flat Universe
Only statistical errors shown

assuming flat Univ. and constant $w$
Exercise 2

- Make the same plot for Worked Example I: plot curves of constant luminosity distance (for several choices of redshift between 0.1 and 1.0) in the plane of $\Omega_\Lambda$ vs. $\Omega_m$, choosing the distance for the model with $\Omega_\Lambda = 0.7$, $\Omega_m = 0.3$ as the fiducial.

- In the same plane, plot the boundary of the region between present acceleration and deceleration.

- Extra credit: in the same plane, plot the boundary of the region that expands forever vs. recollapses.
Cosmic Acceleration

\[ \ddot{a} > 0 \rightarrow \dot{a} = Ha \text{ increases with time} \]

This implies that \( v = Hd \) increases with time: if we could watch the same galaxy over cosmic time, we would see its recession speed increase.

**Exercise 3**: A. Show that \( \ddot{a} > 0 \) implies increasing \( v \).

B. For a galaxy at \( d=100 \) Mpc, if \( H_0=70 \) km/sec/Mpc =constant, what is the increase in its recession speed over a 10-year period? How feasible is it to measure that velocity change?
Bolometric Distance Modulus

- Logarithmic measures of luminosity and flux:

\[ M = -2.5 \log(L) + c_1, \quad m = -2.5 \log(f) + c_2 \]

- Define distance modulus:

\[ \mu \equiv m - M = 2.5 \log(L/f) + c_3 = 2.5 \log(4\pi d_L^2) + c_3 \]

\[ = 5 \log[H_0 d_L(z;\Omega_m,\Omega_{DE},w(z))] - 5 \log H_0 + c_4 \]

\[ = 5 \log[d_L(z;\Omega_m,\Omega_{DE},w(z))]/10pc \]

- For a population of standard candles (fixed \( M \)), measurements of \( \mu \) vs. \( z \), the Hubble diagram, constrain cosmological parameters.
Absolute vs. Relative Distances

- Recall logarithmic measures of luminosity and flux:

  \[ M_i = -2.5 \log(L_i) + c_1, \quad m_i = -2.5 \log(f_i) + c_2 \]

  \[ m_i = 5 \log[H_0 d_L] - 5 \log H_0 + M_i + K(z) + c_4 \]

- If \( M_i \) is known, from measurement of \( m_i \) can infer absolute distance to an object at redshift \( z \), and thereby determine \( H_0 \) (for \( z \ll 1 \), \( d_L = cz/H_0 \))

- If \( M_i \) (and \( H_0 \)) unknown but constant, from measurement of \( m_i \) can infer distance to object at redshift \( z_1 \) relative to object at distance \( z_2 \):

  \[ m_1 - m_2 = 5 \log \left( \frac{d_1}{d_2} \right) + K_1 - K_2 \]

  independent of \( H_0 \)

- Use low-redshift SNe to vertically `anchor' the Hubble diagram, i.e., to determine \( M - 5 \log H_0 \)
Exercise 4

• Plot distance modulus vs redshift ($z=0-1$) for:
  • Flat model with $\Omega_m = 1$
  • Flat model with $\Omega_\Lambda = 0.7$, $\Omega_m = 0.3$
  • Open model with $\Omega_m = 0.3$
    – Plot first linear in $z$, then log $z$.

• Plot the residual of the first two models with respect to the third model
Discovery of Cosmic Acceleration from High-redshift Supernovae (See Saul’s talk)

Type Ia supernovae that exploded when the Universe was 2/3 its present size are ~25% fainter than expected.
Distance and $q_0$

The luminosity distance is given by $d_L(z) = (1+z)a_0r(z)$. Using Eqn. 14 and the result of part (d), to order $z^2$ this gives

$$d_L(z; H_0, q_0) = \frac{z(1+z)}{H_0} \left[ 1 - (1+q_0)\frac{z}{2} \right] = \frac{1}{H_0} \left[ z + z^2 - (1+q_0)\frac{z^2}{2} + \mathcal{O}(z^3) \right]$$

$$= \frac{z}{H_0} \left[ 1 + (1-q_0)\frac{z}{2} \right].$$

(15)

The distance modulus is given by

$$\mu(z; H_0, q_0) = 5 \log_{10} \left( \frac{d_L}{10 \text{ pc}} \right) = 5 \log_{10} \left[ \frac{z}{H_0} \frac{1 + (1-q_0)z/2}{10 \text{ pc}} \right]$$

$$= 5 \log z - 5 \log (H_0 \cdot 10 \text{ pc}) + 5 \log \left[ 1 + \frac{z}{2} (1-q_0) \right].$$

(16)

The last term in Eqn. 16 can be massaged using Stirling’s approximation: for $x \ll 1$, $\ln(1 + x) \simeq x$. Exponentiating and taking the $\log_{10}$ gives $\log_{10} (1 + x) \simeq \log_{10} e^x = x \log_{10} e$, so that

$$5 \log_{10} \left[ 1 + \frac{z}{2} (1-q_0) \right] \simeq \frac{5z}{2} (1-q_0) \log_{10} e = 1.086z(1-q_0).$$

(17)
For a flat Universe, $\Omega_{DE} = 1 - \Omega_m$;

$$q_0 = \frac{\Omega_m}{2} + \frac{(1 - \Omega_m)}{2} (1 + 3w) = \frac{1}{2} + \frac{3w}{2} (1 - \Omega_m) ,$$

so the difference in distance modulus between two flat models with fixed $H_0$ and $\Omega_m$ is

$$\Delta \mu = \frac{3}{2} (1 - \Omega_m)(1.086z)\Delta w = 0.6\Delta w ,$$

where the last expression is evaluated using $\Omega_m = 0.25$ and $z = 0.5$. Since $\sigma_\mu = 0.15$ mag, to determine $w$ to a precision of $\Delta w = 0.1$ requires roughly $\Delta \mu = 0.06 > \sigma_\mu / \sqrt{N} = 0.15 / \sqrt{N}$, or $N > 6$ supernovae. For a precision $\Delta w = 0.01$, we have $\Delta \mu = 0.006$, and we need $N > 600$ supernovae at $z \sim 0.5$. If $\Omega_m$ isn’t exactly known and in the presence of systematic errors, this number of course would be larger.
Volume Element

- Counting a set of objects, e.g., galaxy clusters, with known or predictable spatial number density provides another cosmological test.

- Proper area $dA$ at redshift $z$ and radial coordinate $r$ subtends solid angle $d\Omega$ at the origin given by:

$$dA = a(t_e) r d\theta a(t_e) r \sin \theta d\varphi = a_e^2 r^2 d\Omega = \frac{a_0^2 r^2 d\Omega}{(1 + z_e)^2}$$

- Rate of proper displacement with $z$ along light ray is:

$$d\ell = c dt = \frac{dz}{(1 + z)H(z)}$$

linear depth of sample in redshift interval $(z, z + dz)$

- Proper volume element of sample is then:

$$d^2V_p = dA d\ell = \frac{r^2(z)}{H(z)(1 + z)^3} d\Omega dz$$
Volume Element

• For proper number density of objects $n_p(z)$, the number counts per unit redshift and solid angle are

$$\frac{d^2N}{dzd\Omega} = n_p(z) \frac{d^2V_p}{dzd\Omega} = \frac{n_p(z)r^2(z)}{H(z)(1+z)^3}$$

• Define the comoving number density $n_c(z) = n_p(z)/(1+z)^3$, which is constant if objects are conserved, and comoving volume element $d^2V_c = d^2V_p(1+z)^3$, in which case

$$\frac{d^2N}{dzd\Omega} = n_c(z) \frac{d^2V_c}{dzd\Omega} = \frac{n_c(z)r^2(z)}{H(z)}$$
Volume Element

\[ \left( \frac{d^2V}{dQdz} \right) \times \left( \frac{H_0^3}{z^2} \right) \]

- \( \Omega_M = 0.2, \ w = -1.0 \)
- \( \Omega_M = 0.3, \ w = -1.0 \)
- \( \Omega_M = 0.2, \ w = -1.5 \)
- \( \Omega_M = 0.2, \ w = -0.5 \)

See Steve’s talk