Bootstrapping One-Loop $2 \to n$ Amplitudes

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Stanford Linear Accelerator Center

with

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Loopfest V – June 19th, 2006

References

[1] Z. Bern, L. J. Dixon, D. A. Kosower,
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Phys. Rev. D 72, 125003 (2005) [hep-ph/0505055];
Phys. Rev. D 73, 065013 (2006) [hep-ph/0507005].

[2] CFB, Z. Bern, L. J. Dixon, D. Forde, D. A. Kosower,
hep-ph/0604195.

[3] CFB, Z. Bern, L. J. Dixon, D. Forde, D. A. Kosower,
to appear.

The (In)Famous Les Houches 2005 Wishlist

process wanted at NLO ($V \in \{Z, W, \gamma\}$)	background to
1. $pp \to VV +$ jet	$t\bar{t}H$, new physics
2. $pp \to H + 2$ jets	H production by vector boson fusion (VBF)
3. $pp \to t\bar{t}b\bar{b}$	$t\bar{t}H$
4. $pp \to t\bar{t} + 2$ jets	$t\bar{t}H$
5. $pp \to VVb\bar{b}$	VBF $\to H \to VV$, $t\bar{t}H$, new physics
6. $pp \to VV + 2$ jets	VBF $\to H \to VV$
7. $pp \to V + 3$ jets	new physics
8. $pp \to VVV$	SUSY trilepton

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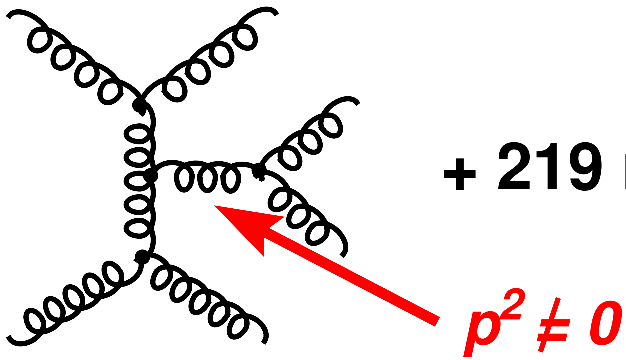
Large number of high-multiplicity processes that need to be computed!
The LHC turns on in 2007!

Feynman Graphs

- Feynman rules are **too general, not optimized, do not take into account all symmetries of the theory**
- Vertices and propagators involve **gauge-dependent off-shell states**
- In real kinematics **no on-shell 3-point vertex**
- **Explosive growth** of number of diagrams/terms



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Time to panic??

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Time to panic?? – No!

➨ **(Semi)Numerical approaches and automatization**

MadEvent, ALPGEN, CompHEP, GRACE, HELAC/PHEGAS, . . .

Kramer, Soper, Nagy; Ellis, Giele, Glover, Zanderighi; Binoth, Ciccolini, Guillet,

Heinrich, Kauer, Pilon, Schubert; Czakon; Anastasiou, Daleo; . . .

➨ **Recursion relations**

Color Ordering, Spinors and Twistors

- Strip color information, only calculate diagrams with cyclic color ordering
  $\Rightarrow$ **36 diagrams instead of 220 for $n = 6$ gluons**
- Use the "right variables" to expose more symmetries - spinor helicity formalism

$$\lambda_i = u_+(p_i) = \frac{1}{2}(1+\gamma_5)u(p_i) \qquad \tilde{\lambda}_i = u_-(p_i) = \frac{1}{2}(1-\gamma_5)u(p_i)$$

$$\langle i\, j\rangle = \langle i^-|j^+\rangle = \bar{u}_-(p_i)u_+(p_j) \quad [i\, j] = \langle i^+|j^-\rangle = \bar{u}_+(p_i)u_-(p_j)$$

Transformation to **Penrose's twistor space** (Fourier transform in $\tilde{\lambda}$)
$\Rightarrow$ **amazingly simple structure of scattering amplitudes**
Witten; Nair; Roiban, Spradlin, Volovich

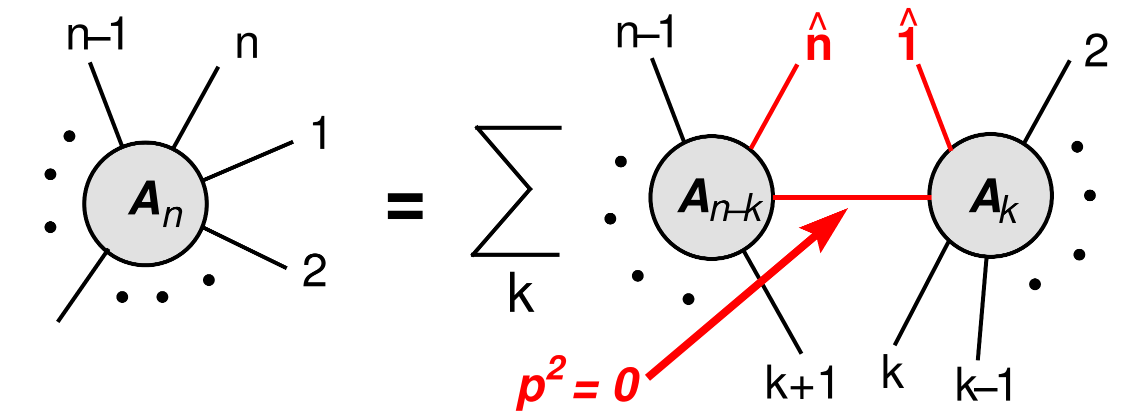
- "Recycle" known amplitudes via **recursion relations**
  Berends, Giele; Mahlon; Cachazo, Svrcek, Witten; Britto, Cachazo, Feng, Witten

Recursion Relations at Tree Level

Complex continue spinors and momenta

(1) $$[j, l\rangle : \quad \tilde{\lambda}_j \to \tilde{\lambda}_j - z\tilde{\lambda}_l \quad \lambda_l \to \lambda_l + z\lambda_j$$

(2) $$p_j^\mu \to p_j^\mu(z) \equiv \hat{p}_j^\mu = p_j^\mu - \frac{z}{2}\langle j^- | \gamma^\mu | l^- \rangle$$
$$p_l^\mu \to p_l^\mu(z) \equiv \hat{p}_l^\mu = p_l^\mu + \frac{z}{2}\langle j^- | \gamma^\mu | l^- \rangle$$



Britto, Cachazo, Feng

Proof at Tree-Level

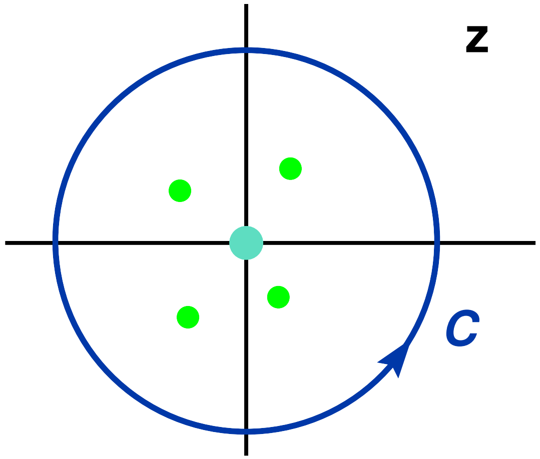
Amplitude function of complex parameter

$$A(z) = A(p_1, \ldots, p_j(z), p_{j+1}, \ldots, p_l(z), \ldots, p_n)$$

If $A(z \to \infty) \to 0$ - Cauchy's theorem

(3) $$\frac{1}{2\pi i} \oint_C \frac{dz}{z} A(z) = 0$$

(4) $$A(0) = - \sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{A(z)}{z}$$



Poles in z correspond to physical factorizations

(5) $$\frac{1}{\hat{P}^2_{l\ldots m}} = \frac{1}{P^2_{l\ldots m} - z \langle j^- | \not{P}_{l\ldots m} | k^- \rangle}$$

Britto, Cachazo, Feng, Witten

On-Shell Recursions

Proof at tree level only relies on Cauchy's theorem and basic factorization properties.

See also: Draggiotis, Kleiss, Lazopoulos, Papadopoulos; Vaman, Yao

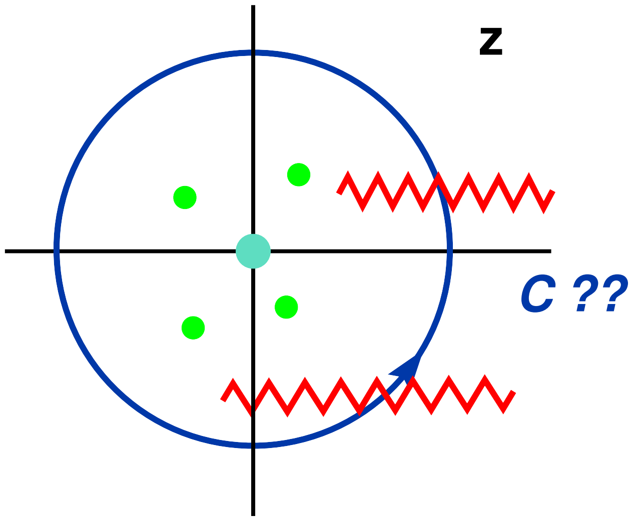
$\Rightarrow$ **Many applications**

- **SUSY - processes with massless fermions** Luo, Wen
- **QCD - QCD is supersymmetric at tree level**
- **Massive scalars and fermions**

  Badger, Glover, Khoze, Svrcek; Forde, Kosower; Schwinn, Weinzierl; Ferrario, Rodrigo, Talavera
- **Higgs (top loop integrated out)** Badger, Dixon, Glover, Khoze
- **Gravity**

  Bedford, Brandhuber, Spence, Travaglini; Cachazo, Svrcek; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager

QCD at One Loop - A Disaster?



- Branch cuts with spurious singularities $\sim \frac{\ln(s_1/s_2)}{(s_1 - s_2)^2}$

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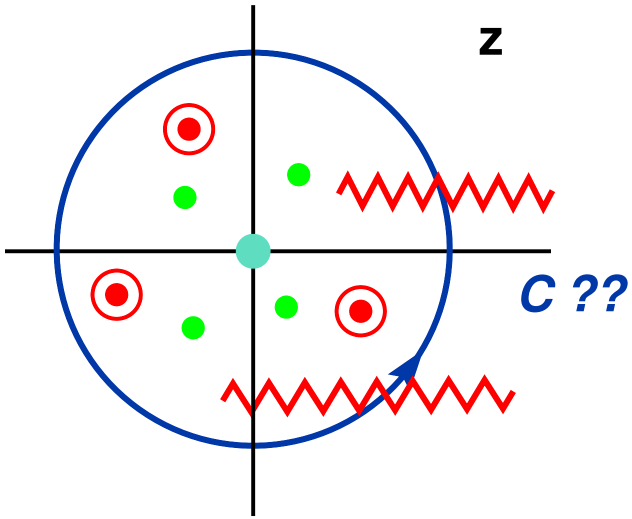

- Branch cuts with spurious singularities $\sim \frac{\ln(s_1/s_2)}{(s_1 - s_2)^2}$
- Double poles $\sim \frac{\langle a\, b\rangle}{[a\, b]^2}$, 'unreal poles' $\sim \frac{\langle a\, b\rangle}{[a\, b]}$, and nonstandard factorizations

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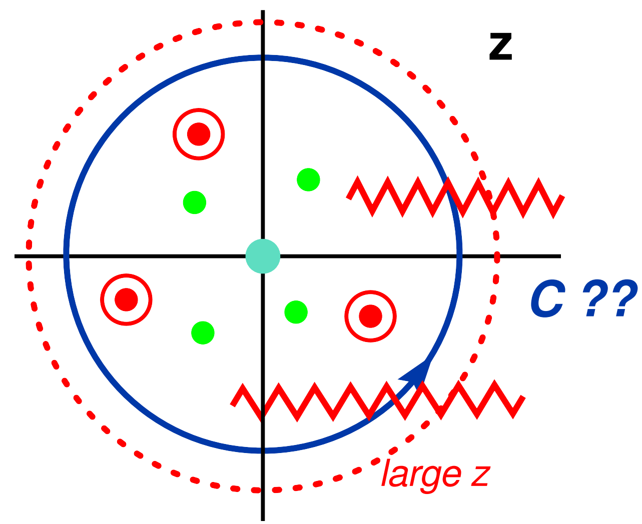

- **Branch cuts with spurious singularities** $\sim \frac{\ln(s_1/s_2)}{(s_1-s_2)^2}$
- **Double poles** $\sim \frac{\langle a\, b\rangle}{[a\, b]^2}$, **'unreal poles'** $\sim \frac{\langle a\, b\rangle}{[a\, b]}$, **and nonstandard factorizations**
- $A(z \to \infty) \neq 0$

On-Shell Bootstrap Method

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Bootstrap model

From Wikipedia, the free encyclopedia

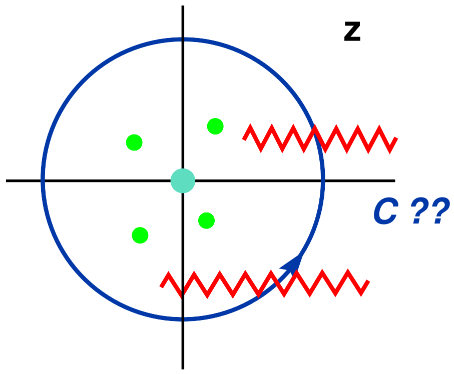
In physics, the term **bootstrap model** is used for the class of theories that assume that very general consistency criteria are sufficient to determine the whole theory completely. In such theories, typically examples of quantum field theory, it is impossible to divide the objects and concepts to elementary and composite ones. See Geoffrey Chew. This strategy turned out to be successful only in the case of two-dimensional conformal field theory where many insights can indeed b derived by this method.

Here: very general consistency criteria

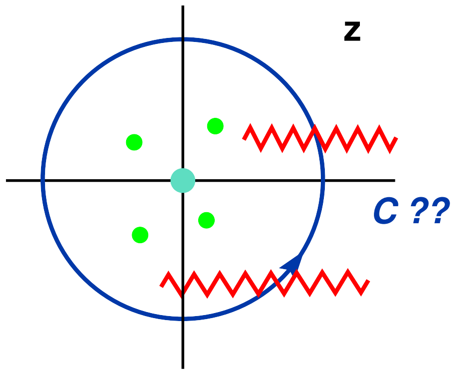
- **Cuts (unitarity)**
- **Poles (factorization)**

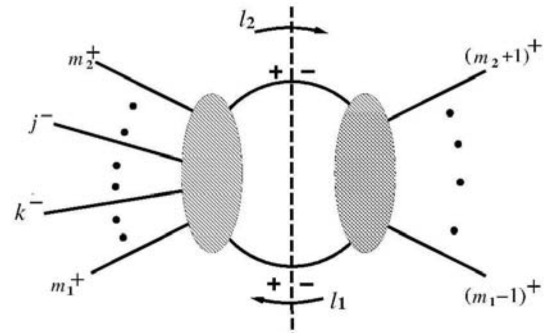
$$A(z) = c_\Gamma \left[C(z) + R(z) \right] \qquad (6)$$

Cut Parts via Unitarity



Cut Parts via Unitarity





$C(0)$ **contains only Li, ln, π^2 – cut-constructible! via (generalized) unitarity**

$$\int d\text{LIPS}(-l_1, l_2) A^{\text{tree}}(-l_1, m_1, .., m_2, l_2) A^{\text{tree}}(-l_2, m_2+1, .., m_1-1, l_1)$$

Trees “recycled” into loops

SUSY: $R = 0$ – fully cut-constructible

Bern, Dixon, Dunbar, Kosower; Bedford, Brandhuber, McNamara, Spence, Travaglini; Quigley, Rozali; Britto, Buchbinder, Cachazo, Feng, Mastrolia; Bern, Bidder, Bjerrum-Bohr, Dixon, Dunbar, Ita, Perkins

Cut Parts and Spurious Singularities

C contains spurious singularities, $\frac{\ln(s_1/s_2)}{(s_1-s_2)^2}$. These cancel in the full amplitude. Reshuffle (= complete cut) terms between C and R,

$$A(z) = c_\Gamma \left[\left(C(z) - \widehat{CR} \right) + \left(R(z) + \widehat{CR} \right) \right]$$

$$\equiv c_\Gamma \left[\quad \hat{C}(z) \quad + \quad \hat{R}(z) \right] \qquad \Bigg| \qquad \frac{1}{2\pi i} \oint_C \frac{dz}{z}$$

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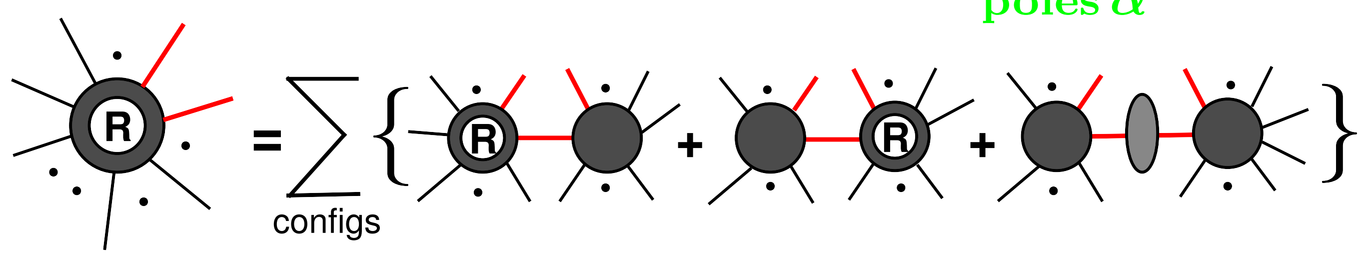
$$\equiv c_\Gamma \left[\hat{C}(z) + \hat{R}(z) \right] \qquad \Bigg| \quad \frac{1}{2\pi i} \oint_C \frac{dz}{z}$$

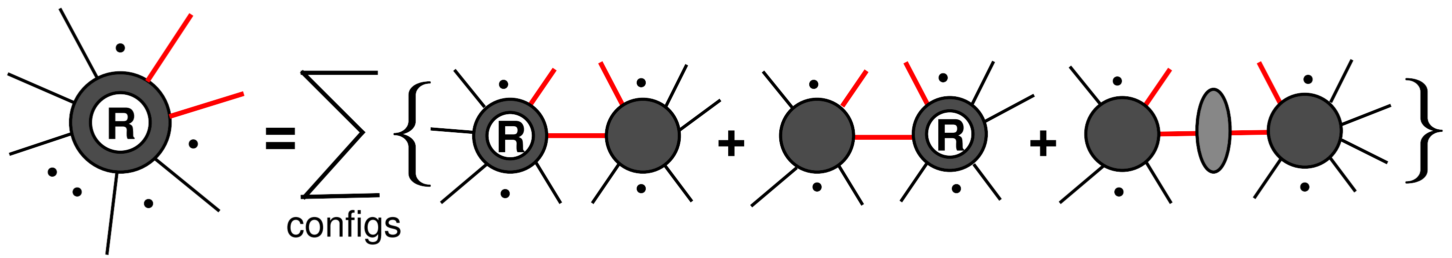
(7) $$A(0) = \mathrm{Inf}\, A + \left(\hat{C}(0) - \mathrm{Inf}\, \hat{C} \right) - \sum_{\text{poles } \alpha} \mathop{\mathrm{Res}}_{z=z_\alpha} \frac{\hat{R}(z)}{z}$$



Loops "recycled" into loops

Bern, Dixon, Kosower

On-Shell Recursion for Rational Parts



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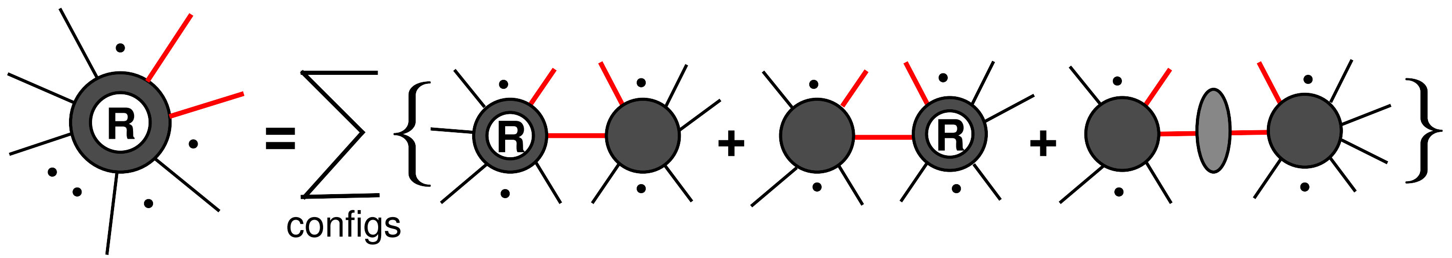


$\widehat{C}$ has a rational part from cut completion, $\widehat{CR}$.
Recursion for rational parts over all poles, including those already included in $\widehat{C}$.
$\Rightarrow$ avoid double counting by subtracting off overlap terms O (not unique)

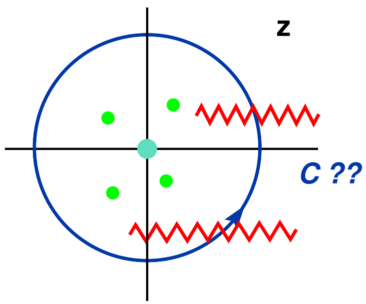
$$R = \widehat{R} + \widehat{CR} \tag{8}$$

$$-\sum_{\text{poles }\alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\widehat{R}(z)}{z} = -\sum_{\text{poles }\alpha} \operatorname*{Res}_{z=z_\alpha} \frac{R(z)}{z} + \sum_{\text{poles }\alpha} \operatorname*{Res}_{z=z_\alpha} \frac{\widehat{CR}(z)}{z}$$

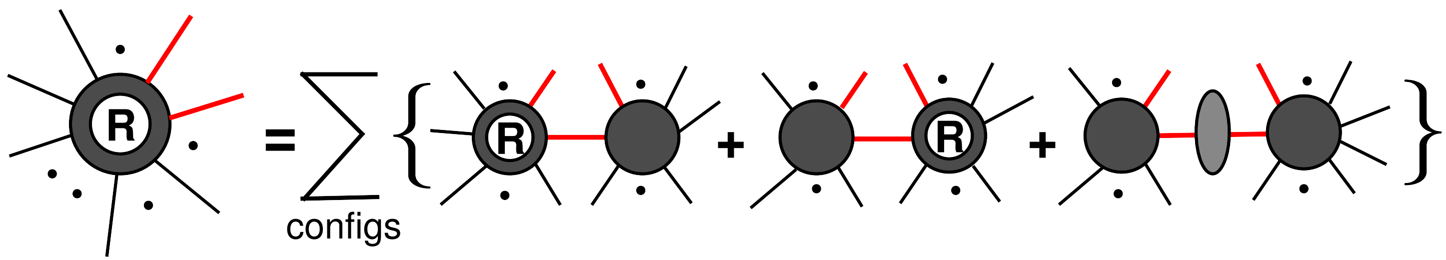
$$\equiv -\sum_{\text{poles }\alpha} \operatorname*{Res}_{z=z_\alpha} \frac{R(z)}{z} + O \tag{9}$$

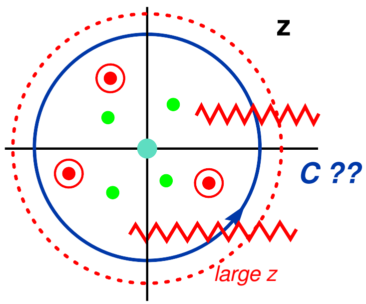
Bern, Dixon, Kosower

Non-Standard Factorizations

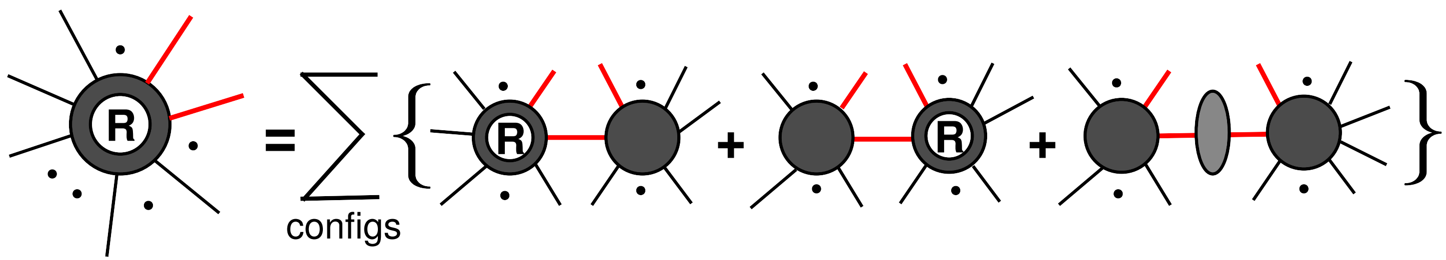


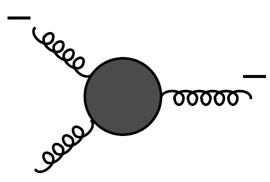
$$A(0) = + c_\Gamma \Big[\widehat{C}(0) - \mathrm{Inf}\,\widehat{C} - \sum_{\text{poles }\alpha} \mathop{\mathrm{Res}}_{z=z_\alpha} \frac{R(z)}{z} + O + \Big]$$



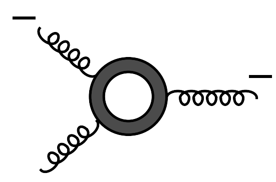


$$A(0) = \textcolor{red}{\mathrm{Inf}\, A} + c_\Gamma \left[\widehat{C}(0) - \mathrm{Inf}\, \widehat{C} - \sum_{\text{poles } \alpha} \operatorname*{Res}_{z=z_\alpha} \frac{R(z)}{z} + O + \textcolor{red}{???} \right]$$

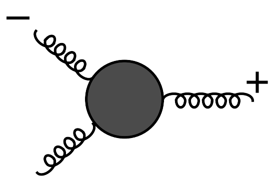




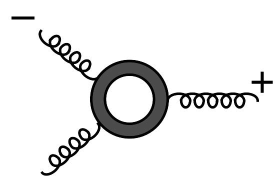
$$= 0$$



$$\sim \frac{\langle a\, b \rangle}{[a\, b]^2}$$



$$\sim \frac{1}{[a\, b]}$$



$$\sim \text{rational} \times \frac{1}{[a\, b]}$$

Factorization properties unclear at one loop.

Large-z Contributions

Can pick shifts to avoid either non-standard factorizations or $z \to \infty$ contributions, but in general not both!

- $[j, l\rangle$ **avoids non-standard factorizations**

$$A(0) = \operatorname*{Inf}_{[j,l\rangle} A + c_\Gamma \left[\widehat{C}(0) - \operatorname*{Inf}_{[j,l\rangle} \widehat{C} + R^{[j,l\rangle}_{\text{recurs}} + O^{[j,l\rangle} \right] \tag{10}$$

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- $[a, b\rangle$ **has no large-parameter contributions**

$$A(0) = c_\Gamma \left[\widehat{C}(0) - \operatorname*{Inf}_{[a,b\rangle} \widehat{C} + R^{[a,b\rangle}_{\text{recurs}} + \text{non-standard channels}^{[a,b\rangle} + O^{[a,b\rangle} \right] \tag{11}$$

The Bootstrap Formalism

Solution $\Rightarrow$ use two shifts!

Extract large-parameter contributions of primary shift from auxiliary relation (11)

$$A(0) = c_\Gamma \left[\widehat{C}(0) - \operatorname*{Inf}_{[a,b\rangle} \widehat{C} + R^{[a,b\rangle}_{\text{recurs}} + O^{[a,b\rangle} + \text{non-std}^{[a,b\rangle} \right] \Bigg|\, \operatorname*{Inf}_{[j,l\rangle}$$

$$\operatorname*{Inf}_{[j,l\rangle} A^{[a,b\rangle} = c_\Gamma \operatorname*{Inf}_{[j,l\rangle} \left[\widehat{C}(0) - \operatorname*{Inf}_{[a,b\rangle} \widehat{C} + R^{[a,b\rangle}_{\text{recurs}} + O^{[a,b\rangle} \right]$$

(12) **if** $\operatorname*{Inf}_{[j,l\rangle} \left[\text{non-standard channels}^{[a,b\rangle} \right] = 0$

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$$\operatorname*{Inf}_{[j,l\rangle} A^{[a,b\rangle} = c_\Gamma \operatorname*{Inf}_{[j,l\rangle} \left[\widehat{C}(0) - \operatorname*{Inf}_{[a,b\rangle} \widehat{C} + R^{[a,b\rangle}_{\text{recurs}} + O^{[a,b\rangle} \right]$$

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The complete bootstrap

$$\boxed{A(0) = \operatorname*{Inf}_{[j,l\rangle} A^{[a,b\rangle} + c_\Gamma \left[\widehat{C}(0) - \operatorname*{Inf}_{[j,l\rangle} \widehat{C} + R^{[j,l\rangle}_{\text{recurs}} + O^{[j,l\rangle}\right]}$$

Passes all nontrivial checks! CFB, Bern, Dixon, Forde, Kosower

Example: $A^{(1)}(1^-,2^-,3^-,4^+,5^+,6^+)$

(13) $$X(1,2,3,4,5,6)\Big|_{\text{flip 1}} \equiv X(3,2,1,6,5,4)$$

$$\widehat{C}_6(1^-,2^-,3^-,4^+,5^+,6^+) = \frac{1}{3c_\Gamma}\,A_{6;1}^{\mathcal{N}=1}(1^-,2^-,3^-,4^+,5^+,6^+)$$

(14) $$+\ \frac{2}{9}A_6^{\text{tree}}(1^-,2^-,3^-,4^+,5^+,6^+) + \hat{C}_6^a + \hat{C}_6^a\Big|_{\text{flip 1}}$$

$$\hat{C}_6^a =$$

$$\frac{i}{3}\left[\frac{\langle 1\,2\rangle\langle 2\,3\rangle[2\,4]\langle 1^-|\,(3+4)\,|2^-\rangle\Big[\langle 3^-|\,42\,|1^+\rangle P_{234}^2 - \langle 3^-|\,2(3+4)\,|1^+\rangle P_{34}^2\Big]}{\langle 3\,4\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,[2\,3]\,\langle 5^-|\,(3+4)\,|2^-\rangle}\,\frac{\mathsf{L}_2\!\left(\frac{-P_{234}^2}{-P_{34}^2}\right)}{(P_{34}^2)^3}\right.$$

$$\left. + \frac{\langle 3\,5\rangle[4\,5][5\,6]\langle 5^-|\,(1+2)\,|6^-\rangle\Big[\langle 3^-|\,(5-4)\,|6^-\rangle P_{345}^2 + \langle 3^-|\,(4+5)\,|6^-\rangle P_{34}^2\Big]}{\langle 4\,5\rangle\,[1\,2]\,[1\,6]\,\langle 5^-|\,(3+4)\,|2^-\rangle}\,\frac{\mathsf{L}_2\!\left(\frac{-P_{345}^2}{-P_{34}^2}\right)}{(P_{34}^2)^3}\right]$$

$$\mathsf{L}_2(r) = \frac{\ln(r)-(r-1/r)/2}{(1-r)^3}$$

Bern, Bjerrum-Bohr, Dunbar, Ita

Example: $A^{(1)}(1^-,2^-,3^-,4^+,5^+,6^+)$

(13) $$X(1,2,3,4,5,6)\Big|_{\text{flip }1} \equiv X(3,2,1,6,5,4)$$

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$$\left.+\ \frac{\langle 3\,5\rangle[4\,5][5\,6]\langle 5^-|\,(1+2)\,|6^-\rangle\left[\langle 3^-|\,(5-4)\,|6^-\rangle P_{345}^2 + \langle 3^-|\,(4+5)\,|6^-\rangle P_{34}^2\right]}{\langle 4\,5\rangle\,[1\,2]\,[1\,6]\,\langle 5^-|\,(3+4)\,|2^-\rangle}\,\frac{\mathsf{L}_2\left(\frac{-P_{345}^2}{-P_{34}^2}\right)}{(P_{34}^2)^3}\right]$$

$$\mathsf{L}_2(r) = \frac{\ln(r)-(r-1/r)/2}{(1-r)^3}$$

Bern, Bjerrum-Bohr, Dunbar, Ita

Shift $[1,2\rangle$

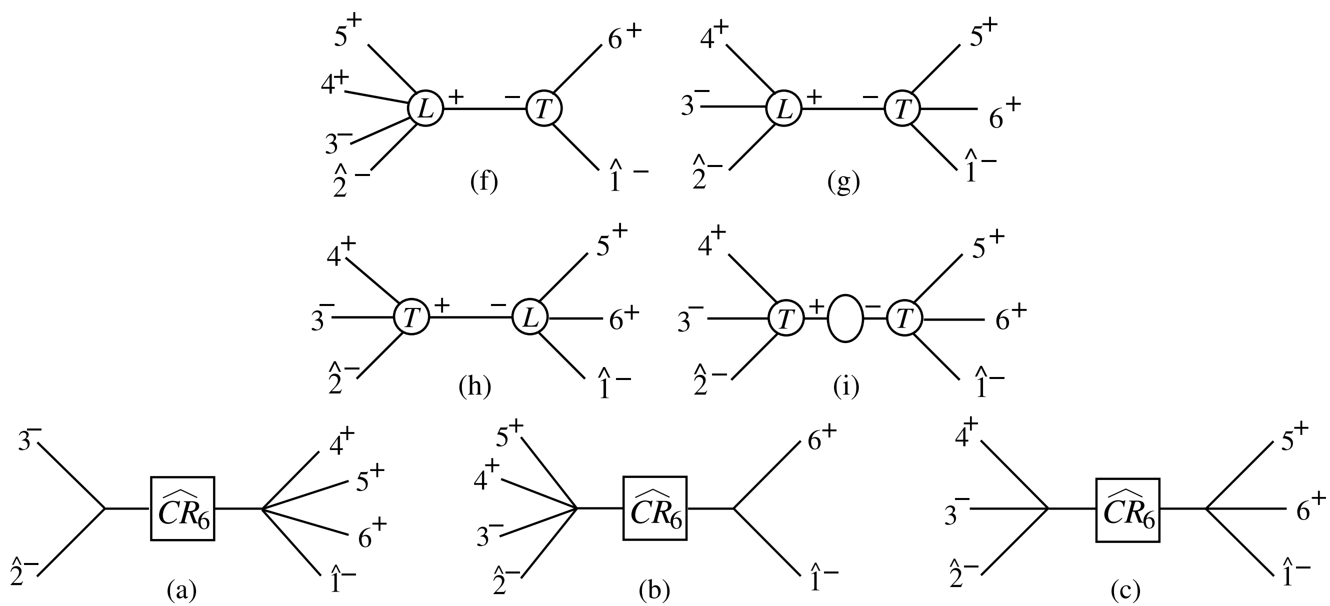
$$\Rightarrow \mathrm{Inf}_{[1,2\rangle}\, \widehat{C}_6 = \lim_{z\to\infty} \widehat{C}_6$$

Recursive and overlap contributions in channels

$$(15)\qquad P_{61}^2 \;\to\; P_{61}^2 - z\,\langle 1^-|\,\not{P}_{61}\,|2^-\rangle$$

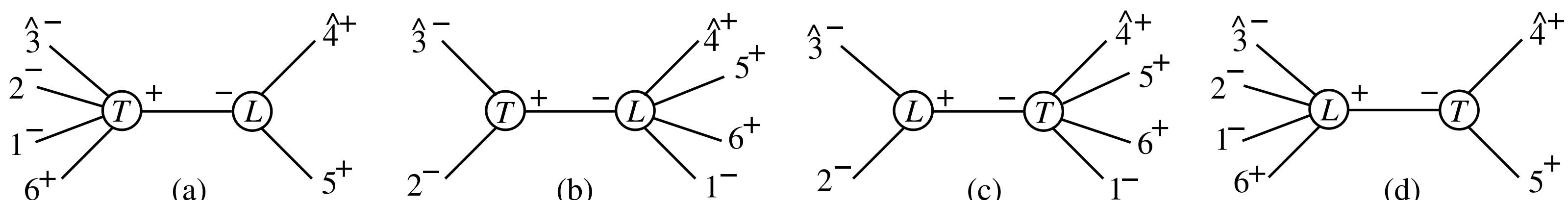
$$(16)\qquad P_{23}^2 \;\to\; P_{23}^2 + z\,\langle 1^-|\,\not{P}_{23}\,|2^-\rangle$$

$$(17)\qquad P_{234}^2 \;\to\; P_{234}^2 + z\,\langle 1^-|\,\not{P}_{234}\,|2^-\rangle$$



Example: $A^{(1)}(1^-,2^-,3^-,4^+,5^+,6^+)$ contd.

Auxiliary recursion relation for $\mathrm{Inf}_{[1,2\rangle}\, A$



$$\mathop{\mathrm{Inf}}_{[1,2\rangle} A_{6;1}(1^-,2^-,3^-,4^+,5^+,6^+) =$$

$$\mathop{\mathrm{Inf}}_{[1,2\rangle} A_{5;1}(1^-,2^-,\hat{3}^-,\hat{K}_{45}^+,6^+)\,\frac{i}{P_{45}^2}\,A_3^{\mathrm{tree}}(-\hat{K}_{45}^-,\hat{4}^+,5^+) \tag{18}$$

$$\widehat{R}_6 = \widehat{R}_6^a + \widehat{R}_6^a\Big|_{\text{flip 1}} \tag{19}$$

$$\begin{aligned}
\widehat{R}_6^a = \frac{i}{6}\frac{1}{[2\,3]\,\langle 5\,6\rangle\,\langle 5^-|\,(3+4)\,|2^-\rangle}\Bigg\{ & -\frac{[4\,6]^3\,[2\,5]\,\langle 5\,6\rangle}{[1\,2]\,[3\,4]\,[6\,1]} - \frac{\langle 1\,3\rangle^3\,\langle 2\,5\rangle\,[2\,3]}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 6\,1\rangle} \\
& + \frac{\langle 1^-|\,(2+3)\,|4^-\rangle^2}{[3\,4]\,\langle 6\,1\rangle}\left(\frac{\langle 1^-|\,2\,|4^-\rangle - \langle 1^-|\,5\,|4^-\rangle}{P_{234}^2} + \frac{\langle 1\,3\rangle}{\langle 3\,4\rangle} - \frac{[4\,6]}{[6\,1]}\right) \\
& - \frac{\langle 1\,3\rangle^2(3\,\langle 1^-|\,2\,|4^-\rangle + \langle 1^-|\,3\,|4^-\rangle)}{\langle 3\,4\rangle\,\langle 6\,1\rangle} \\
& + \frac{[4\,6]^2(3\,\langle 1^-|\,5\,|4^-\rangle + \langle 1^-|\,6\,|4^-\rangle)}{[3\,4]\,[6\,1]}\Bigg\}
\end{aligned}$$

$\Rightarrow$ **all-n solution**

CFB, Bern, Dixon, Forde, Kosower

Results

- ☑ **All-multiplicity formulae for $(+ + \ldots +)$, $(- + \ldots +)$ one-loop gluon amplitudes (also with a fermion pair)**

  Bern, Dixon, Kosower

- ☑ **All-multiplicity formulae for $(+ \ldots + - + \ldots + - + \ldots +)$ one-loop gluon amplitudes**

  Forde, Kosower; CFB, Bern, Dixon, Forde, Kosower

- ☑ **All-multiplicity formulae for $(- - - + \ldots +)$ one-loop gluon amplitudes**

  CFB, Bern, Dixon, Forde, Kosower

- ☑ **Some all-multiplicity results for parts of Higgs plus gluons (and fermion pair) at NNLO (effective theory - top loop integrated out)**

  CFB, Del Duca, Dixon

All of the above $\ll \infty$ pages

- ☑ **Working algorithm for all other configurations of one-loop gluon amplitudes!**

  CFB, Bern, Dixon, Forde, Kosower

To-Do List

- ❑ Understand complex factorization at one loop and beyond + connection to Lagrangian?
- ❑ Higher loops?
- ❑ Massive partons (external fermions, scalars, . . .)
- ❑ Automatization
- ❑ Attack the wishlists...

Summary

"One of the most remarkable discoveries in elementary particle physics has been that of the existence of the complex plane."

in J. Schwinger, "Particles, Sources, and Fields", Vol. I.