$e^+e^- \to 3j$ at NNLO:
Results for all QED-type Colour Factors

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Loopfest 2006 – SLAC

$e^+e^- \to 3$ jets and event shapes

Classical QCD observable

- testing ground for QCD: perturbation theory, power corrections and logarithmic resummation
- precision measurement of strong coupling constant α_s
- current error on α_s from jet observables dominated by theoretical uncertainty: S. Bethke, 2006

$$\alpha_s(M_Z) \quad = \quad 0.121 \pm 0.001(\text{experiment}) \pm 0.005(\text{theory})$$

- theoretical uncertainty largely from missing higher orders
- current status: NLO plus NLL resummation

$e^+e^- \to 3$ jets and event shapes

Event shape variables

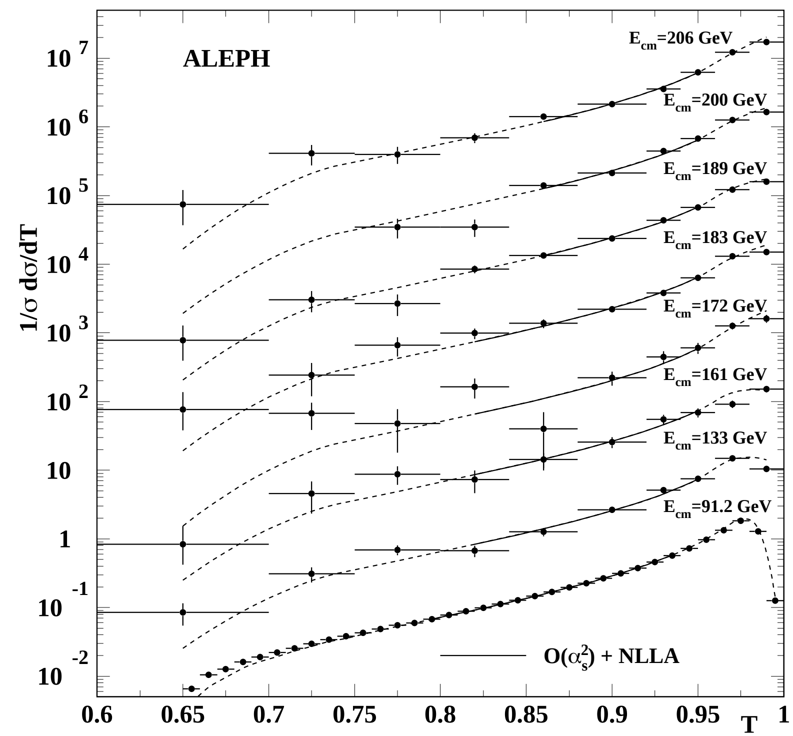
assign a number x to a set of final state momenta: $\{p\}_i \to x$

e.g. Thrust in e^+e^-

$$T = \max_{\vec{n}} \frac{\sum_{i=1}^{n} |\vec{p_i} \cdot \vec{n}|}{\sum_{i=1}^{n} |\vec{p_i}|}$$

limiting values:

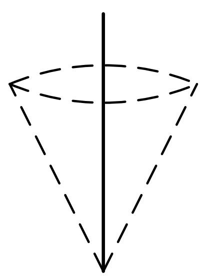
- back-to-back (two-jet) limit: $T = 1$
- spherical limit: $T = 1/2$



Jet Observables

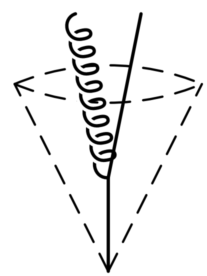
Theoretically:

- Partons are combined into jets using the same jet algorithm (recombination procedure) as in experiment



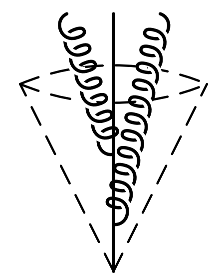
LO

each parton forms 1 jet on its own



NLO

2 partons in 1 jet, 1 parton experimentally unresolved



NNLO

3 partons in 1 jet, 2 partons experimentally unresolved

Current state-of-the-art: NLO

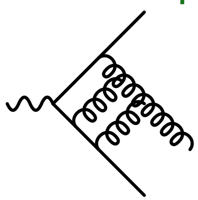
Need for NNLO:

- reduce error on α_s
- better matching of parton level and hadron level jet algorithm

Ingredients to NNLO 3-jet

- Two-loop matrix elements

$|\mathcal{M}|^2_{\text{2-loop},3}$ partons

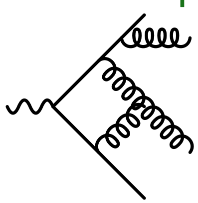


explicit infrared poles from loop integrals

L. Garland, N. Glover, A. Koukoutsakis, E. Remiddi, TG;
S. Moch, P. Uwer, S. Weinzierl

- One-loop matrix elements

$|\mathcal{M}|^2_{\text{1-loop},4}$ partons

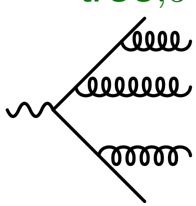


explicit infrared poles from loop integral and
implicit infrared poles due to single unresolved radiation

Z. Bern, L. Dixon, D. Kosower, S. Weinzierl;
J. Campbell, D.J. Miller, E.W.N. Glover

- Tree level matrix elements

$|\mathcal{M}|^2_{\text{tree},5}$ partons



implicit infrared poles due to double unresolved radiation

K. Hagiwara, D. Zeppenfeld;
F.A. Berends, W.T. Giele, H. Kuijf;
N. Falck, D. Graudenz, G. Kramer

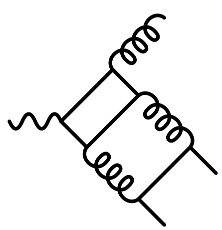
Infrared Poles cancel in the sum

Colour structure of NNLO 3-jet

Decomposition into leading and subleading colour terms

$$\sigma_{NNLO} = (N^2-1)\Bigg[N^2\, A_{NNLO} + B_{NNLO} + \frac{1}{N^2}\, C_{NNLO} + N N_F\, D_{NNLO}$$
$$+\frac{N_F}{N}\, E_{NNLO} + N_F^2\, F_{NNLO} + N_{F,\gamma}\left(\frac{4}{N} - N\right)\, G_{NNLO}\Bigg]$$

- last term: closed quark loop coupling to vector boson, numerically tiny



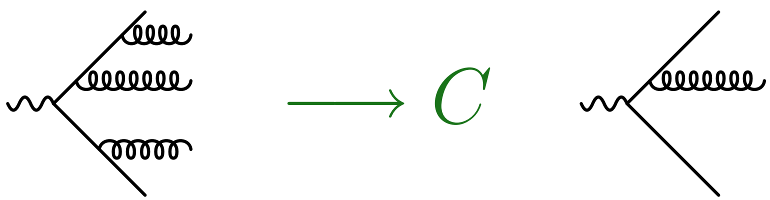
$$N_{F,\gamma} = \frac{\left(\sum_q e_q\right)^2}{\sum_q e_q^2}$$

- most subleading colour: C_{NNLO}, E_{NNLO}, F_{NNLO}, (G_{NNLO})
  QED-type contributions: gluons $\longrightarrow$ photons
- simplest term: F_{NNLO}, only 3 parton and 4 parton contributions

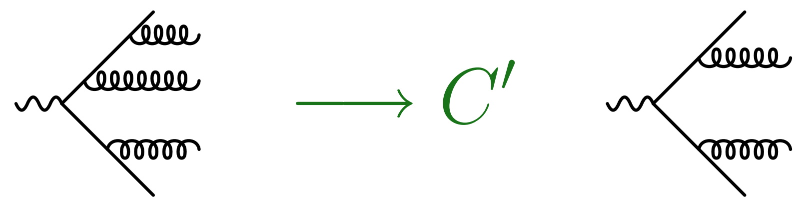
Real Radiation at NNLO

Infrared subtraction terms

$m+2$ partons $\to$ m jets:



$m+2 \to m+1$ pseudopartons $\to$ m jets:



- Double unresolved configurations:
  - triple collinear
  - double single collinear
  - soft/collinear
  - double soft

- Single unresolved configurations:
  - collinear
  - soft

J. Campbell, E.W.N. Glover; S. Catani, M. Grazzini

Issue: find subtraction functions which

- approximate full $m+2$ matrix element in all singular limits
- are sufficiently simple to be integrated analytically

NLO Subtraction

Structure of NLO m-jet cross section (subtraction formalism):
Z. Kunszt, D. Soper

$$d\sigma_{NLO} = \int_{d\Phi_{m+1}} \left(d\sigma^R_{NLO} - d\sigma^S_{NLO}\right) + \left[\int_{d\Phi_{m+1}} d\sigma^S_{NLO} + \int_{d\Phi_m} d\sigma^V_{NLO}\right]$$

- $d\sigma^S_{NLO}$: local counter term for $d\sigma^R_{NLO}$
- $d\sigma^R_{NLO} - d\sigma^S_{NLO}$: free of divergences, can be integrated numerically

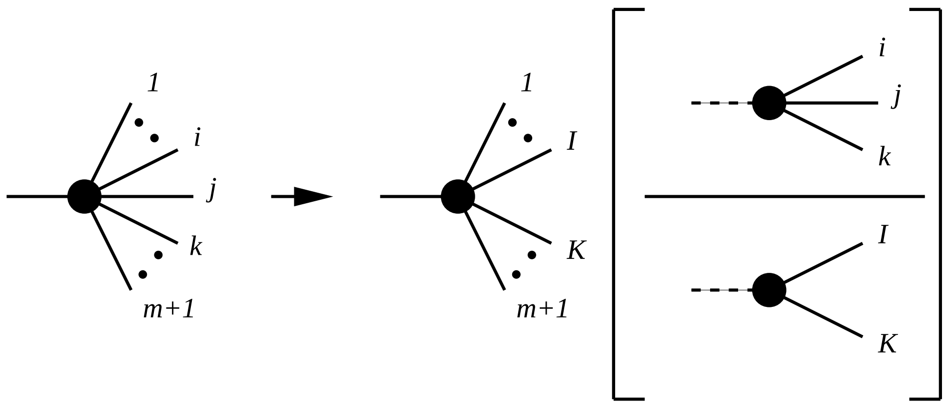
General methods at NLO

- Dipole subtraction
  S. Catani, M. Seymour; NNLO: S. Weinzierl
- $\mathcal{E}$-prescription
  S. Frixione, Z. Kunszt, A. Signer; NNLO: S. Frixione, M. Grazzini
- Antenna subtraction
  D. Kosower; J. Campbell, M. Cullen, N. Glover; NNLO: this talk

NLO Antenna Subtraction

Building block of $\mathrm{d}\sigma^{S}_{NLO}$: NLO-Antenna function X^0_{ijk}

Contains all singularities of parton j emitted between partons i and k



$$X^0_{ijk} = S_{ijk,IK}\frac{|M^0_{ijk}|^2}{|M^0_{IK}|^2}$$

$$\mathrm{d}\Phi_{X_{ijk}} = \frac{\mathrm{d}\Phi_3}{P_2}$$

Phase space factorisation

$$\mathrm{d}\Phi_{m+1}(p_1,\ldots,p_{m+1};q) = \mathrm{d}\Phi_m(p_1,\ldots,\tilde{p}_I,\tilde{p}_K,\ldots,p_{m+1};q)\cdot \mathrm{d}\Phi_{X_{ijk}}(p_i,p_j,p_k;\tilde{p}_I+\tilde{p}_K)$$

Integrated subtraction term (analytically)

$$|\mathcal{M}_m|^2\, J_m^{(m)}\, \mathrm{d}\Phi_m \int \mathrm{d}\Phi_{X_{ijk}}\, X^0_{ijk} \sim |\mathcal{M}_m|^2\, J_m^{(m)}\, \mathrm{d}\Phi_m \int \mathrm{d}\Phi_3 |M^0_{ijk}|^2$$

can be combined with $\mathrm{d}\sigma^{V}_{NLO}$

NNLO Subtraction

Structure of NNLO m-jet cross section:

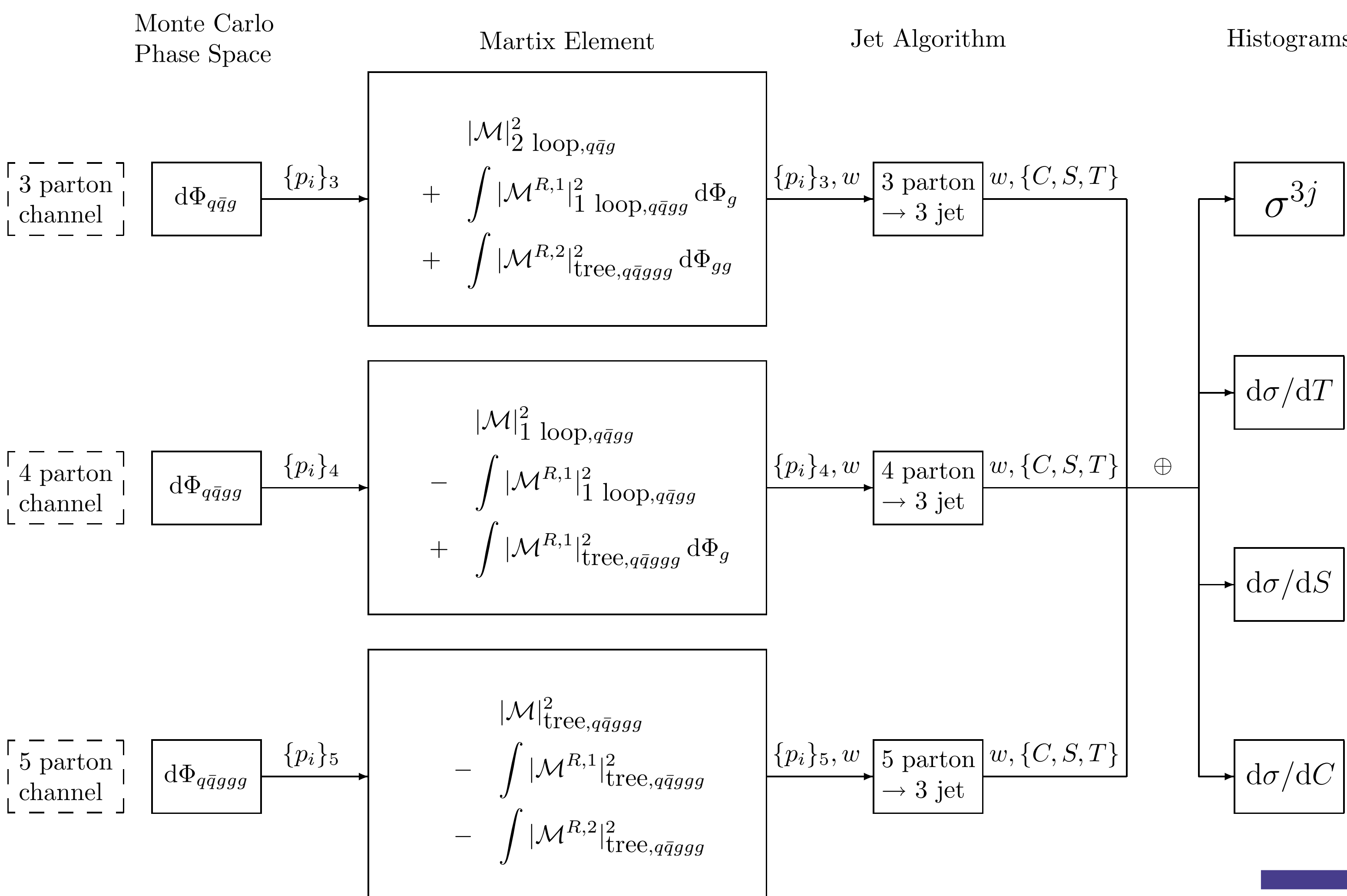
$$
\begin{aligned}
\mathrm{d}\sigma_{NNLO} \quad = \quad & \int_{\mathrm{d}\Phi_{m+2}} \left(\mathrm{d}\sigma^{R}_{NNLO} - \mathrm{d}\sigma^{S}_{NNLO}\right) \\
& + \int_{\mathrm{d}\Phi_{m+1}} \left(\mathrm{d}\sigma^{V,1}_{NNLO} - \mathrm{d}\sigma^{VS,1}_{NNLO}\right) \\
& + \int_{\mathrm{d}\Phi_{m}} \mathrm{d}\sigma^{V,2}_{NNLO} + \int_{\mathrm{d}\Phi_{m+2}} \mathrm{d}\sigma^{S}_{NNLO} + \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\sigma^{VS,1}_{NNLO} \,,
\end{aligned}
$$

- $\mathrm{d}\sigma^{S}_{NNLO}$: real radiation subtraction term for $\mathrm{d}\sigma^{R}_{NNLO}$
- $\mathrm{d}\sigma^{VS,1}_{NNLO}$: one-loop virtual subtraction term for $\mathrm{d}\sigma^{V,1}_{NNLO}$
- $\mathrm{d}\sigma^{V,2}_{NNLO}$: two-loop virtual corrections

Each line above is finite numerically and free of infrared ϵ-poles $\longrightarrow$ numerical programme

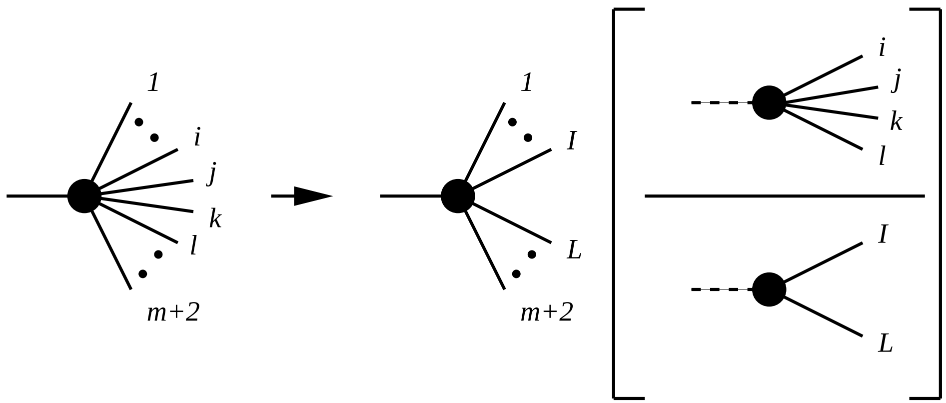
NNLO Subtraction

Structure of $e^+e^- \to 3$ jets program:



Double Real Subtraction

Two colour-connected unresolved partons



$$X^0_{ijkl} = S_{ijkl,IL} \frac{|M^0_{ijkl}|^2}{|M^0_{IL}|^2}$$

$$\mathrm{d}\Phi_{X_{ijkl}} = \frac{\mathrm{d}\Phi_4}{P_2}$$

Phase space factorisation

$$\mathrm{d}\Phi_{m+2}(p_1,\ldots,p_{m+2};q) = \mathrm{d}\Phi_m(p_1,\ldots,\tilde{p}_I,\tilde{p}_L,\ldots,p_{m+2};q)\cdot\mathrm{d}\Phi_{X_{ijkl}}(p_i,p_j,p_k,p_l;\tilde{p}_I+\tilde{p}_L)$$

Integrated subtraction term (analytically)

$$|\mathcal{M}_m|^2\, J_m^{(m)}\ \mathrm{d}\Phi_m \int \mathrm{d}\Phi_{X_{ijkl}}\ X^0_{ijkl} \sim |\mathcal{M}_m|^2\, J_m^{(m)}\ \mathrm{d}\Phi_m \int \mathrm{d}\Phi_4 |M^0_{ijkl}|^2$$

Four-particle inclusive phase space integrals are known

A. Gehrmann-De Ridder, G. Heinrich, TG

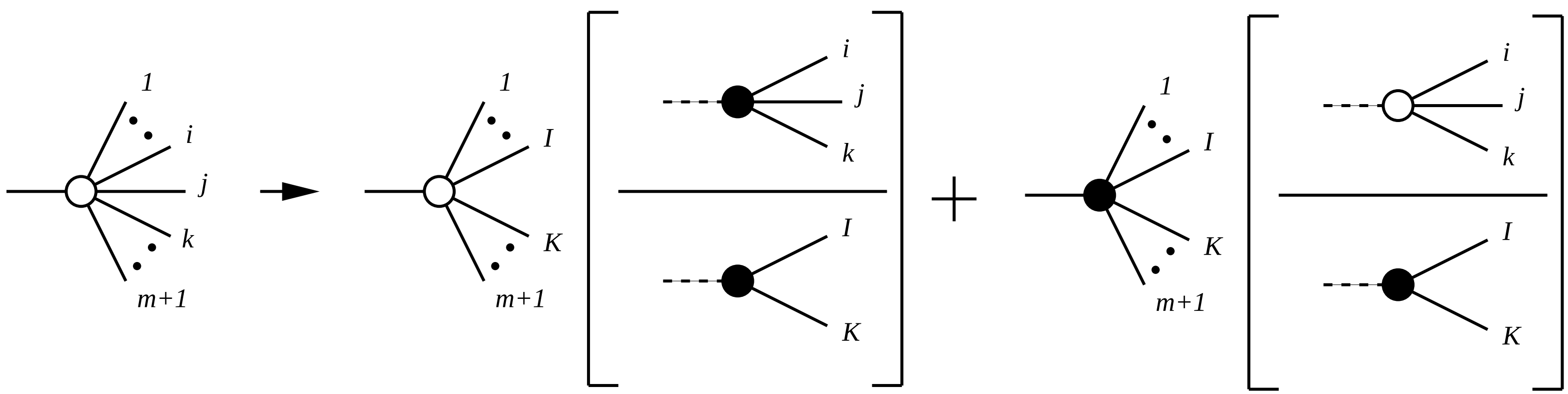
One-loop Real Subtraction

Single unresolved limit of one-loop amplitudes

$$Loop_{m+1} \stackrel{j\ unresolved}{\longrightarrow} Split_{tree} \times Loop_m + Split_{loop} \times Tree_m$$

Z. Bern, L.D. Dixon, D. Dunbar, D. Kosower; S. Catani, M. Grazzini; D. Kosower, P. Uwer
Z. Bern, V. Del Duca, W.B. Kilgore, C.R. Schmidt
Z. Bern, L.D. Dixon, D. Kosower; S. Badger, E.W.N. Glover

Accordingly: $Split_{tree} \to X^0_{ijk}$, $Split_{loop} \to X^1_{ijk}$



$$X^1_{ijk} = S_{ijk,IK} \frac{|\mathcal{M}^1_{ijk}|^2}{|\mathcal{M}^0_{IK}|^2} - X^0_{ijk} \frac{|\mathcal{M}^1_{IK}|^2}{|\mathcal{M}^0_{IK}|^2}$$

Colour-ordered antenna functions

Antenna Functions

- colour-ordered pair of hard partons (radiators) with radiation in between
  - hard quark-antiquark pair
  - hard quark-gluon pair
  - hard gluon-gluon pair
- three-parton antenna $\longrightarrow$ one unresolved parton
- four-parton antenna $\longrightarrow$ two unresolved partons
- can be at tree level or at one loop
- all three-parton and four-parton antenna functions can be derived from physical matrix elements, normalised to two-parton matrix elements

  A. Gehrmann-De Ridder, N. Glover, TG
  - $q\bar{q}$ from $\gamma^* \to q\bar{q} + X$
  - qg from $\chi \to \tilde{g}g + X$
  - gg from $H \to gg + X$

QED-type contributions

Partonic channels $1/N^2$: only quark-antiquark as hard radiator

- $\gamma^* \to q\bar{q}ggg$ and $\gamma^* \to q\bar{q}q\bar{q}g$ at tree-level
  - four-parton antenna functions: $\tilde{A}_4^0(q,g,g,\bar{q})$ and $C_4^0(q,\bar{q},q,\bar{q})$
- $\gamma^* \to q\bar{q}gg$ and $\gamma^* \to q\bar{q}q\bar{q}$ at one loop
  - three-parton one-loop antenna function: $\tilde{A}_3^1(q,g,\bar{q})$
- $\gamma^* \to q\bar{q}g$ at two loops

Partonic channels N_F/N: quark-antiquark or quark-gluon as hard radiator

- $\gamma^* \to q\bar{q}q'\bar{q}'g$ at tree-level
  - four-parton antenna functions: $B_4^0(q,q',\bar{q}',\bar{q})$ and $\tilde{E}_4^0(q,q',\bar{q}',g)$
- $\gamma^* \to q\bar{q}gg$ and $\gamma^* \to q\bar{q}q'\bar{q}'$ at one loop
  - three-parton one-loop antenna functions: $\hat{A}_3^1(q,g,\bar{q})$ and $\tilde{E}_3^1(q,q',\bar{q}')$
- $\gamma^* \to q\bar{q}g$ at two loops

Partonic channels N_F^2: only quark-antiquark as hard radiator

- $\gamma^* \to q\bar{q}q'\bar{q}'$ at one loop
  - three-parton one-loop antenna function: $\hat{E}_3^1(q,q',\bar{q}')$
- $\gamma^* \to q\bar{q}g$ at two loops

Numerical Implementation

Parton-level event generator

Starting point $e^+e^- \to 4$ jets at NLO (EERAD2: J. Campbell, M. Cullen, N. Glover)

- contains already 4-parton and 5-parton matrix elements
- is based on NLO antenna subtraction

modified phase space generation: matrix element

- decompose phase space into wedges, according to relative size of invariants
- each wedge contributes only to some unresolved regions
- angular correlations cancel out (at least to large part) by combining several wedges

modified phase space generation: antenna subtraction terms

- uniform mapping of antenna phase space (D. Kosower)
- requires ordering of unresolved emissions

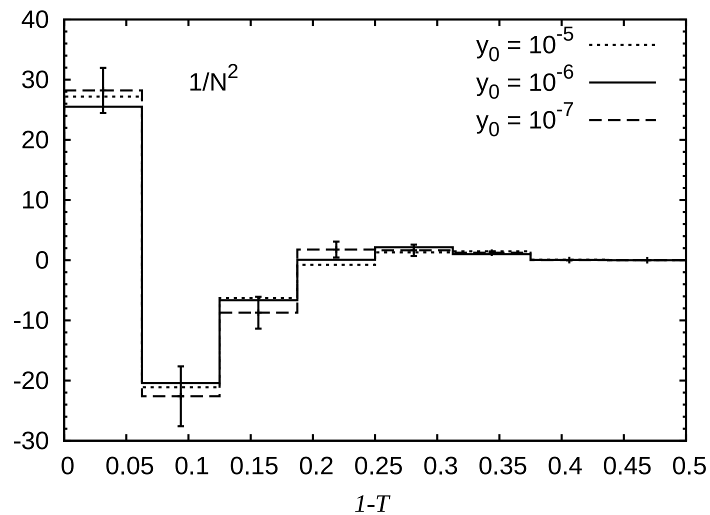
checks

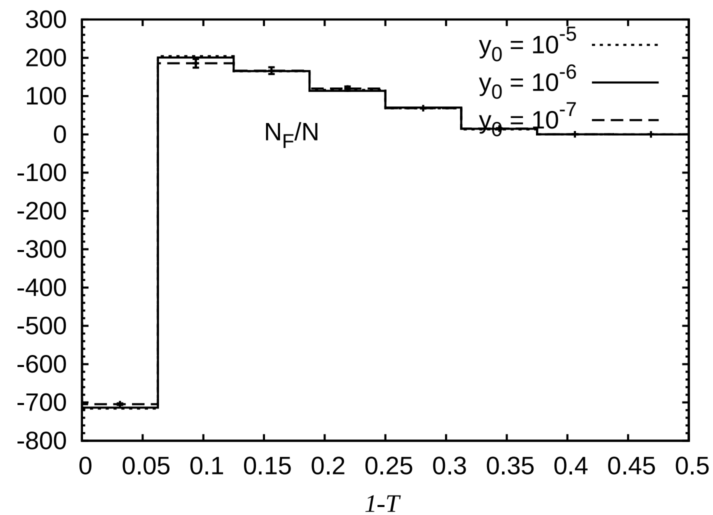
- independence on phase space cut y_0
- local cancellations along phase space trajectories
- distributions in raw phase space variables

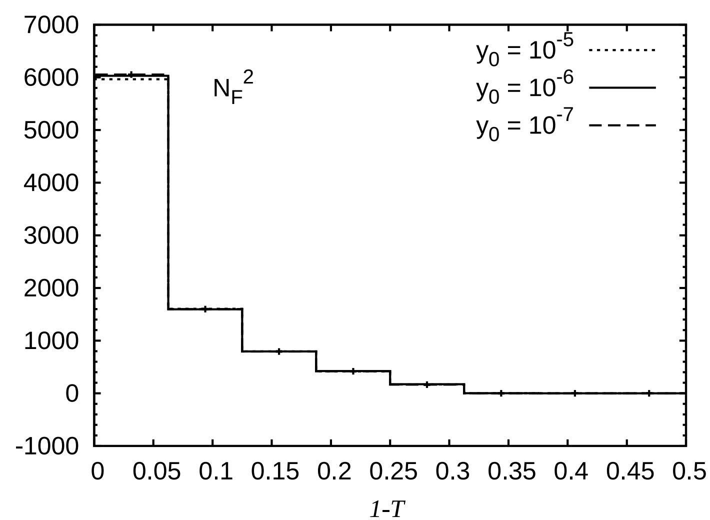
Results

QED-type contributions to $e^+e^- \to 3$ jets

A. Gehrmann-De Ridder, N. Glover, G. Heinrich, TG







Thrust distribution

- independent of phase space cut y_0
- CPU time about 1 day on 2.8 GHz Athlon

Summary and Outlook

Antenna subtraction at NNLO

- building blocks of subtraction terms: 3 and 4 parton antenna functions
- antenna functions are derived from physical $|\mathcal{M}|^2$
- subtraction terms:
  - approximate correctly the full $|\mathcal{M}|^2$ (double real, one-loop/real)
  - do not oversubtract
  - can be integrated analytically

NNLO corrections to $e^+e^- \to 3$ jets

- completed QED-type colour factors: $1/N^2$, N_F/N, N_F^2
- remaining colour factors N^2, N^0, $N_F\,N$ in progress

Application to hadron colliders

- two-loop scattering matrix elements known
- antenna subtraction must also account for initial state singularities: same antenna functions, but different phase space