A semi-numerical approach to one-loop multi-leg amplitudes

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LoopFest V, SLAC, 21.06.06

The start of the LHC is round the corner !

- extracting the signal and studying properties of the Higgs boson/new particles will be an experimental challenge
- theoretical predictions should be as precise as possible

event rates at the LHC

process	events per second
QCD jets $E_T > 150$ GeV	100
$W \to e\nu$	15
$t\bar{t}$	1
Higgs, $m_H = 130$ GeV	0.02
gluinos, $m = 1$ TeV	0.001

$\Rightarrow$ enormous backgrounds !

NLO predictions mandatory for reliable estimates,

especially in the presence of kinematic cuts

NLO wishlist for LHC (Les Houches workshop 05)

process ($V \in \{Z, W, \gamma\}$)	background to
1. $pp \to V\,V$ jet	$t\bar{t}H$, new physics
2. $pp \to t\bar{t}\,b\bar{b}$	$t\bar{t}H$
3. $pp \to t\bar{t} + 2$ jets	$t\bar{t}H$
4. $pp \to V\,V\,b\bar{b}$	VBF$\to H \to VV$, $t\bar{t}H$, new physics
5. $pp \to V\,V + 2$ jets	VBF$\to H \to VV$
6. $pp \to V + 3$ jets	various new physics signatures
7. $pp \to V\,V\,V$	SUSY trilepton

NLO predictions for multi-particle processes

- most of the relevant background processes involve multi-particle final states with several mass scales

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- main difficulties are numerical instabilities and enormous sizes of expressions ($\Rightarrow$ long CPU times, debugging complex)
- virtual amplitudes are the stumbling block for automatisation
- not addressed in this talk:

  – real radiation

  (general algorithms for arbitrary N well established at NLO)

  – combination with parton shower

  $\rightarrow$ talks of S. Frixione, Z. Nagy & Tuesday afternoon

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  $pp \to 3$ jets

  $pp \to V +$ 2 jets

  $pp \to \gamma\gamma$ jet

  $pp \to t\bar{t}H, b\bar{b}H$

  $pp \to t\bar{t}$ jet

  $p\bar{p} \to W\, b\bar{b}$

  Beenakker, Bern, Brandenburg, Campbell, Dawson, De Florian, Del Duca, Dittmaier, Dixon, R.K. Ellis, Febres Cordero, Frixione, Giele, Glover, Kilgore, Kosower, Krämer, Kunszt, Maltoni, Miller, Nagy, Orr, Plümper, Reina, Signer, Spira, Trocsanyi, Uwer, Wackeroth, Weinzierl, Zerwas, . . .

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  $2 \to 3$ for e^+e^- collisions:

  $e^+e^- \to 4$ jets

  $e^+e^- \to \nu\bar{\nu}H$
  $e^+e^- \to e^+e^-H$
  $e^+e^- \to \nu\bar{\nu}\gamma$
  $e^+e^- \to t\bar{t}H$
  $e^+e^- \to ZHH, ZZH$
  $\gamma\gamma \to t\bar{t}H$

  Bern, Dixon, Kosower, Weinzierl, Signer, Nagy, Trocsanyi, Campbell, Cullen, Glover, Miller, Denner, Dittmaier, Roth, Weber, Bélanger, Boudjema, Fujimoto, Ishikawa, Kaneko, Kato, Kurihara, Shimizu, Yasui, Jegerlehner, Tarasov, Chen, Ren-You, Wen-Gan, Hui, Yan-Bin, Hong-Sheng, Pei-Yun, . . .

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- $2 \to 4$:
  - $e^+e^- \to 4$ fermions (complete EW corrections)
    Denner, Dittmaier, Roth, Wieders 02/05
  - $e^+e^- \to HH\nu\bar{\nu}$
    GRACE group (Boudjema et al.) 10/05

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  - $e^+e^- \to HH\nu\bar{\nu}$
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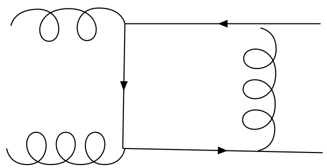
  - no complete σ^{NLO} for $2 \to 4$ processes at the LHC known !
  - 6-gluon amplitude
    Berger, Bern, Del Duca, Dixon, Dunbar, Forde, Kosower;
    Ellis, Giele Zanderighi

multi-particle processes at NLO

main problem with "conventional" approaches beyond $N = 4$ external legs:

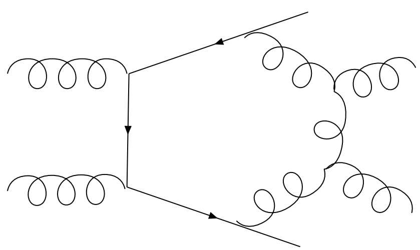
- tensor reduction à la Passarino-Veltman leads to severe numerical instabilities (vanishing inverse Gram determinants)
- analytical representation in terms of a fixed set of Logs, Dilogs not convenient in all phase space regions (e.g. large cancellations between Dilogs) "Polylogarithmic mess" (L. Dixon)
- sheer complexity of expressions

example



4-point integrals (boxes):

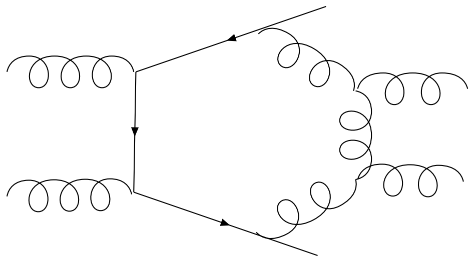
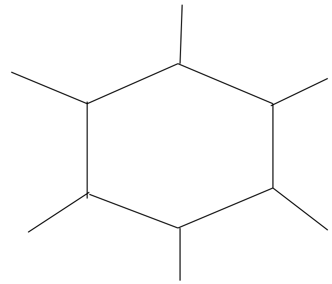
- known analytically
- invariants s, t, m_q



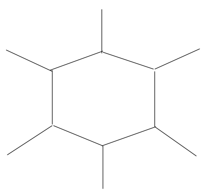
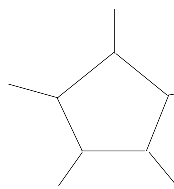
6-point integrals (hexagons):

- 10 invariants,
  one non-linear constraint
- for analytic representation:
  reduce to integrals with less than 5 legs

algebraic reduction

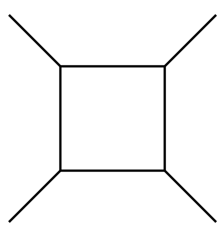
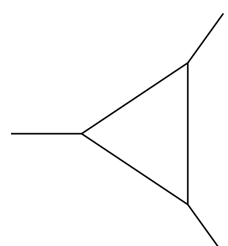
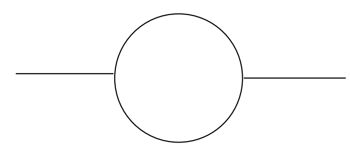
 $\rightarrow$  + integrals with less legs
from reduction of tensor rank and
number of legs at the same time

non-trivial tensor structure   scalar 6-point function

 $= \sum_{i=1}^{6} b_i$  i $\ldots$

reduction until set of basis integrals is reached

basis integrals: 2-, 3- and 4-point functions $\Rightarrow$ known

$\mathcal{A} = C_4$  $+ C_3$  $+ C_2$ 

new approaches

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- semi-numerical methods (tensor reduction and/or integration of loop integrals partly numerically)
  $\longrightarrow$ see also talks of S. Dittmaier, W. Giele
- fully numerical methods
  $\longrightarrow$ talks of A. Daleo, E. de Doncker; see also D. Soper, Z. Nagy

Golem

our approach:

semi-numerical method implemented in program `GOLEM`

(General One-Loop Evaluator for Matrix elements)

[Binoth, Guffanti, Guillet, GH, Karg, Kauer, Pilon, Reiter, . . .]

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main features:

- use algebraic reduction until singularities can be easily isolated (dim. reg.)
- inverse Gram determinants can be completely avoided by choosing a convenient set of (non-scalar) basis integrals ("stop reduction before it generates problems")
- valid for massless and massive particles and for (in principle) arbitrary number of legs

The algorithm

diagram generation (QGRAF, FeynArts)

$$\Downarrow$$

$A = \sum_i C_i^{\mu_1 \ldots \mu_r} \, I_{\mu_1 \ldots \mu_r}$ (FORM)

$$\Downarrow$$

reduction to basis integrals

$\swarrow$ semi-numerically $\qquad$ $\searrow$ algebraically

$A = \sum$ Lorentz tensors $\times$ form factors

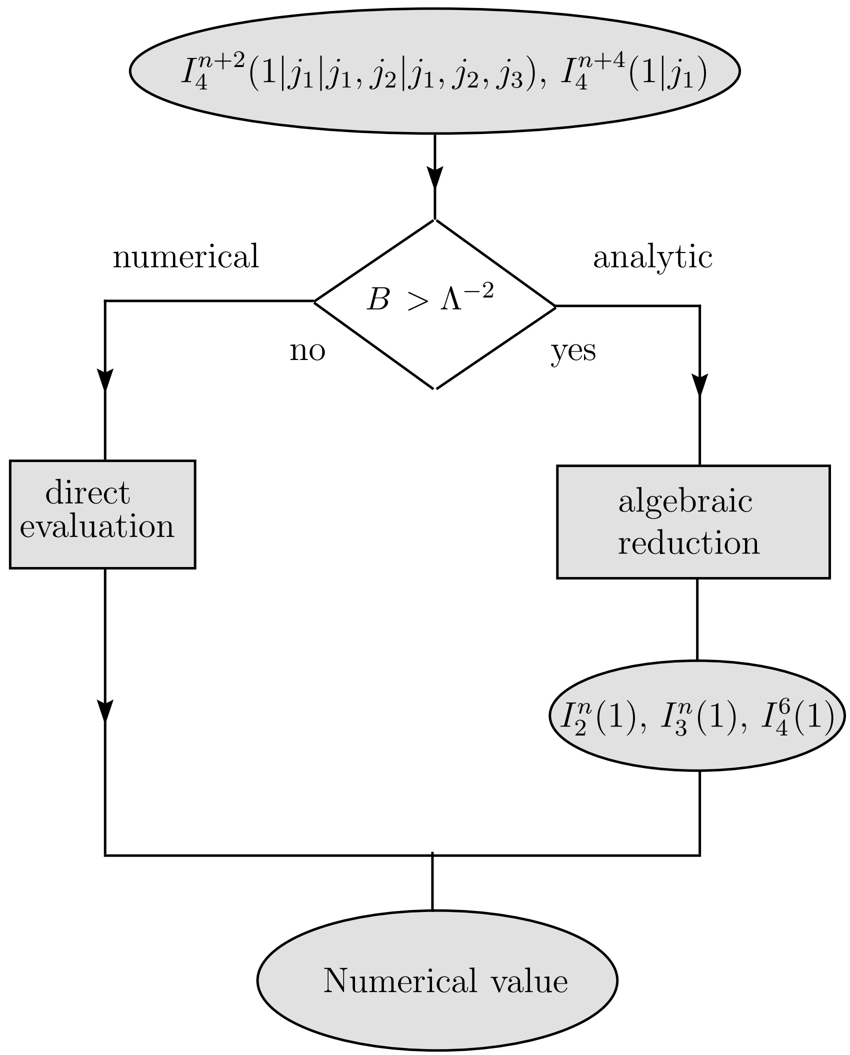
form factors $= \sum_i K_i \, I_i^{\mathrm{master}}$

$\swarrow$ numerical evaluation $\qquad$ $\searrow$ reduction to analytic functions

$\searrow \quad \swarrow$

phase space integration

treatment of basis integrals, example N=4



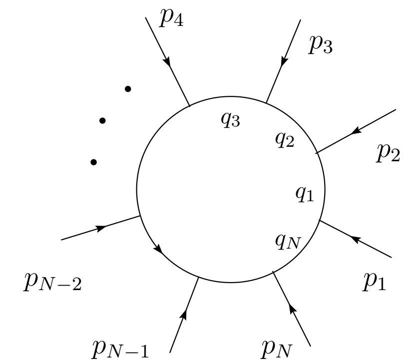
special features

- manifestly shift invariant formulation
- basis integrals can contain Feynman parameters in the numerator ("tensor integrals")
  $\Rightarrow$ way to avoid inverse Gram determinants
- IR poles only in 3-point (and 2-point) functions
  $\Rightarrow$ easily isolated
- proven that higher dimensional integrals I_N^{n+2m} $(m > 0)$ are absent for all $N \geq 5$
- form factors: algebraic simplifications already built in

manifest shift invariance

$$I_N^{n,\,\mu_1\ldots\mu_r}(a_1,\ldots,a_r) = \int \frac{d^n k}{i\,\pi^{n/2}}\,\frac{q_{a_1}^{\mu_1}\;\cdots\;q_{a_r}^{\mu_r}}{(q_1^2-m_1^2+i\delta)\ldots(q_N^2-m_N^2+i\delta)}$$

$$q_i = k + r_i\,,\;\; r_j - r_{j-1} = p_j\,,\;\; \sum_{j=1}^{N} p_j = 0$$



advantages of this representation:

- combinations $q_i = k + r_i$ appear naturally (e.g. fermion propagators)
- manifestly shift invariant formulation by Lorentz tensors defined in terms of difference vectors $\Delta_{i\,j}^{\mu} = r_i^{\mu} - r_j^{\mu}$ and $g^{\mu\nu}$

Lorentz structure and form factors

$$I_N^{n,\,\mu_1\ldots\mu_r}(a_1,\ldots,a_r;\;S) =$$

$$\sum_{l_1\cdots l_r\in S}\left[\Delta^{\cdot}_{l_1\cdot}\cdots\Delta^{\cdot}_{l_r\cdot}\right]^{\{\mu_1\cdots\mu_r\}}_{\{a_1\cdots a_r\}}\;A^{N,r}_{l_1\ldots,l_r}(S)$$

$$+\sum_{l_1\cdots l_{r-2}\in S}\left[g^{\cdot\cdot}\Delta^{\cdot}_{l_1\cdot}\cdots\Delta^{\cdot}_{l_{r-2}\cdot}\right]^{\{\mu_1\cdots\mu_r\}}_{\{a_1\cdots a_r\}}\;B^{N,r}_{l_1\ldots,l_{r-2}}(S)$$

$$+\sum_{l_1\cdots l_{r-4}\in S}\left[g^{\cdot\cdot}g^{\cdot\cdot}\Delta^{\cdot}_{l_1\cdot}\cdots\Delta^{\cdot}_{l_{r-4}\cdot}\right]^{\{\mu_1\cdots\mu_r\}}_{\{a_1\cdots a_r\}}\;C^{N,r}_{l_1\ldots,l_{r-4}}(S)$$

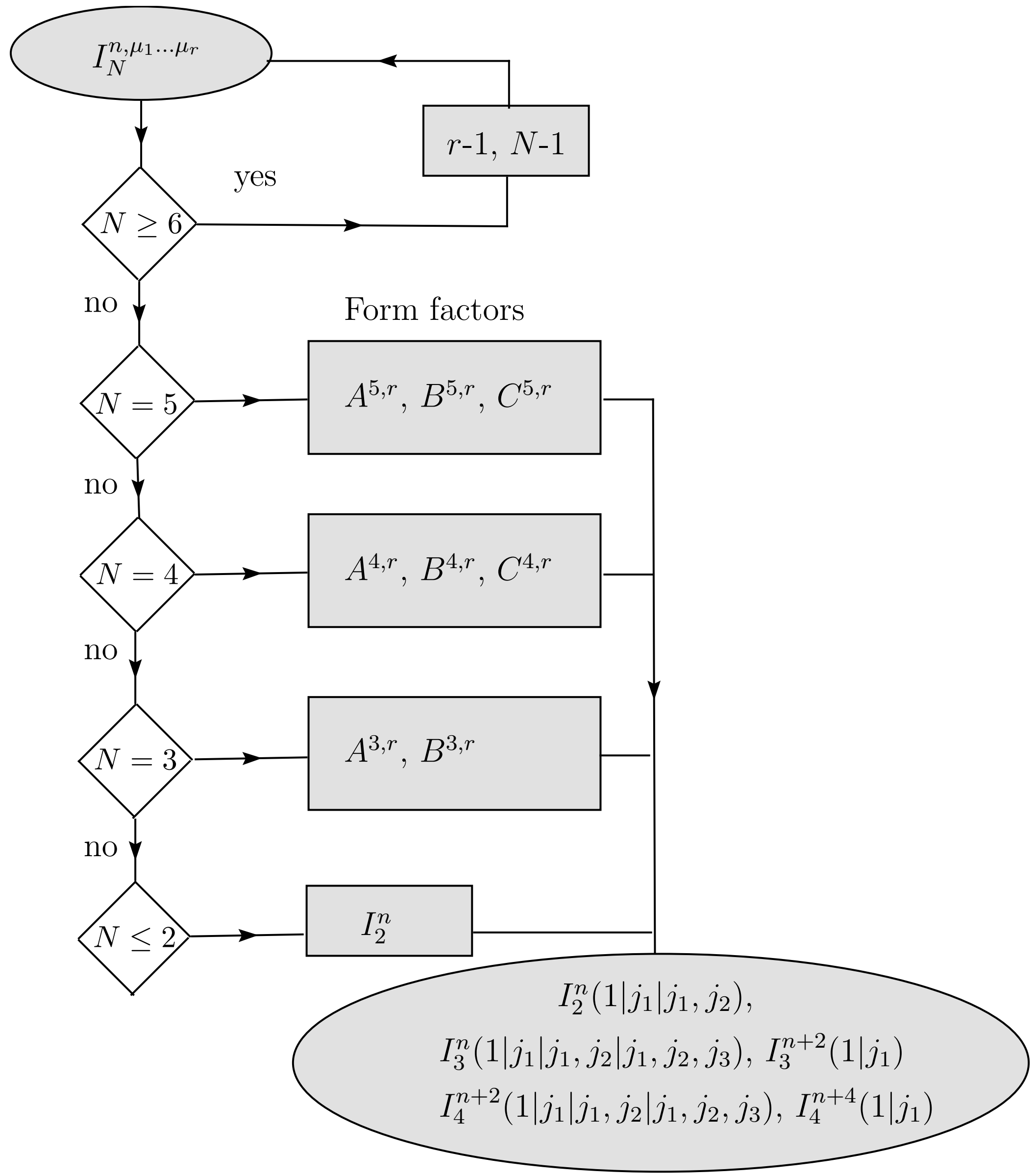
important: more than two metric tensors $g^{\mu\nu}$ never occur !

in a gauge where rank $\leq$ nb of legs

reason: for $N \geq 6$:

$$I_N^{n,\mu_1\ldots\mu_r}(a_1,\ldots,a_r;S) \;=\; -\sum_{j\in S}\mathcal{C}^{\mu_1}_{ja_1}\;I_{N-1}^{n,\mu_2\ldots\mu_r}(a_2,\ldots,a_r;S\setminus\{j\})$$

schematic overview



basis integrals

$$I_3^n(j_1,\ldots,j_r) = -\Gamma\left(3-\frac{n}{2}\right)\int_0^1 \prod_{i=1}^{3} dz_i\,\delta(1-\sum_{l=1}^{3} z_l)\,\frac{z_{j_1}\ldots z_{j_r}}{(-\frac{1}{2}\,z\cdot\mathcal{S}\cdot z - i\delta)^{3-n/2}}$$

$$I_3^{n+2}(j_1) = -\Gamma\left(2-\frac{n}{2}\right)\int_0^1 \prod_{i=1}^{3} dz_i\,\delta(1-\sum_{l=1}^{3} z_l)\,\frac{z_{j_1}}{(-\frac{1}{2}\,z\cdot\mathcal{S}\cdot z - i\delta)^{2-n/2}}$$

$$I_4^{n+2}(j_1,\ldots,j_r) = \Gamma\left(3-\frac{n}{2}\right)\int_0^1 \prod_{i=1}^{4} dz_i\,\delta(1-\sum_{l=1}^{4} z_l)\,\frac{z_{j_1}\ldots z_{j_r}}{(-\frac{1}{2}\,z\cdot\mathcal{S}\cdot z - i\delta)^{3-n/2}}$$

$$I_4^{n+4}(j_1) = \Gamma\left(2-\frac{n}{2}\right)\int_0^1 \prod_{i=1}^{4} dz_i\,\delta(1-\sum_{l=1}^{4} z_l)\,\frac{z_{j_1}}{(-\frac{1}{2}\,z\cdot\mathcal{S}\cdot z - i\delta)^{2-n/2}}$$

and scalar $I_3^n, I_3^{n+2}, I_4^{n+2}, I_4^{n+4}$

important: $r_{\rm max} = 3$

alternatives for evaluation:

1. direct numerical evaluation (contour deformation)

2. further algebraic reduction to scalar integrals

algebraic reduction: example N=4, r=1

algebraic reduction re-introduces inverse Gram determinants $\sim 1/B$

$$I_4^{n+2}(l;S) = \frac{1}{B} \left\{ b_l \, I_4^{n+2}(S) + \frac{1}{2} \sum_{j \in S} \mathcal{S}^{-1}{}_{jl} \, I_3^n(S \setminus \{j\}) - \frac{1}{2} \sum_{j \in S \setminus \{l\}} b_j \, I_3^n(l; S \setminus \{j\}) \right\}$$

$$B = \mathrm{Det}(G) / \mathrm{Det}(\mathcal{S}) \, (-1)^{N+1}$$

$$\mathcal{S}_{ij} = (r_i - r_j)^2 - m_i^2 - m_j^2 \qquad G_{ij} = 2 \, r_i \cdot r_j$$

algebraic reduction: example N=4, r=1

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\begin{aligned}
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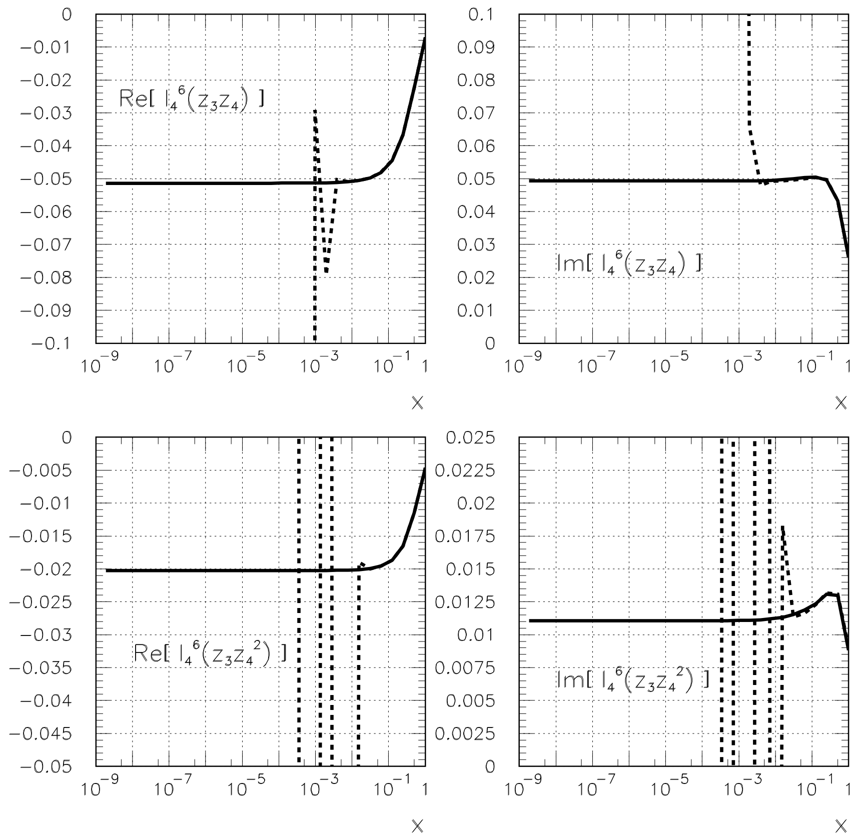
B &= \mathrm{Det}(G)/\mathrm{Det}(\mathcal{S})\,(-1)^{N+1} \\
\mathcal{S}_{ij} &= (r_i - r_j)^2 - m_i^2 - m_j^2 \qquad G_{ij} = 2\, r_i\cdot r_j
\end{aligned}
$$

- $1/B$ does not pose a problem in most regions of phase space
- allows fast evaluation
- $\Rightarrow$ use numerical method only in regions where $1/B$ becomes small

numerical behaviour



$I_4^6(z_3 z_4)$ and $I_4^6(z_3 z_4^2)$

(occur in the reduction of a rank 6 hexagon)

exceptional kinematics for $x \longrightarrow 0$

solid line: numerical evaluation, only partial algebraic reduction

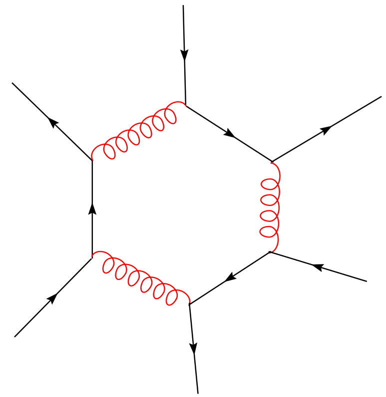
dashed: numerical behaviour of analytic representation after full algebraic reduction

Applications

- $gg \longrightarrow W^* W^* \longrightarrow l\,\nu\,l'\,\nu'$ Binoth, Ciccolini, Kauer, Krämer
- $gg \longrightarrow H\,H\,H$ Binoth, Karg

Applications

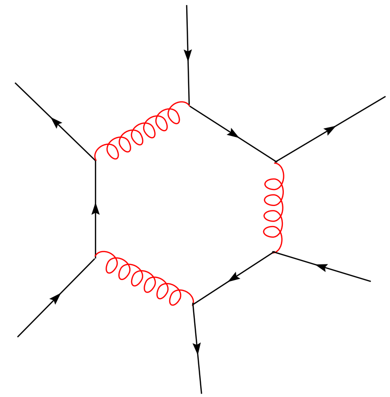
- $q\bar{q} \longrightarrow q\bar{q}\,q\bar{q}$



$$\mathcal{A}^{++++++}(k_1,\ldots,k_6) = \frac{g^6}{(4\pi)^{n/2}}\,\frac{1}{s}\,\left\{\frac{A_2}{\epsilon^2} + \frac{A_1}{\epsilon} + A_0 + \mathcal{O}(\epsilon)\right\}$$

Applications

- $q\bar{q} \to q\bar{q}\, q\bar{q}$



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sample point

k	k^0	k^x	k^y	k^z
k_1	0.5	0	0	0.5
k_2	0.5	0	0	-0.5
k_3	0.364669	0.125147	-0.069442	-0.335410
k_4	0.327318	-0.247566	0.098117	0.190319
k_5	0.208422	0.155873	0.062586	0.123395
k_6	0.099591	-0.033454	-0.091261	0.021696

$q\bar{q} \longrightarrow q\bar{q}\, q\bar{q}$

hexagon:

Re(P_2)	Im(P_2)	Re(P_1)	Im(P_1)	Re(P_0)	Im(P_0)
-3.27889	-0.95055	-10.7288	-15.5340	1.16693	-59.117

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- numerical evaluation of basis integrals ($B < \Lambda^{-2}$) for about 1% of phase space points ($\Lambda \sim 20 - 50$ GeV)
- numerical evaluation slower but
  - not a problem for 1% of phase space
  - these regions often excluded by experimental cuts

Summary and Outlook

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  $\longrightarrow$ important tools for the LHC

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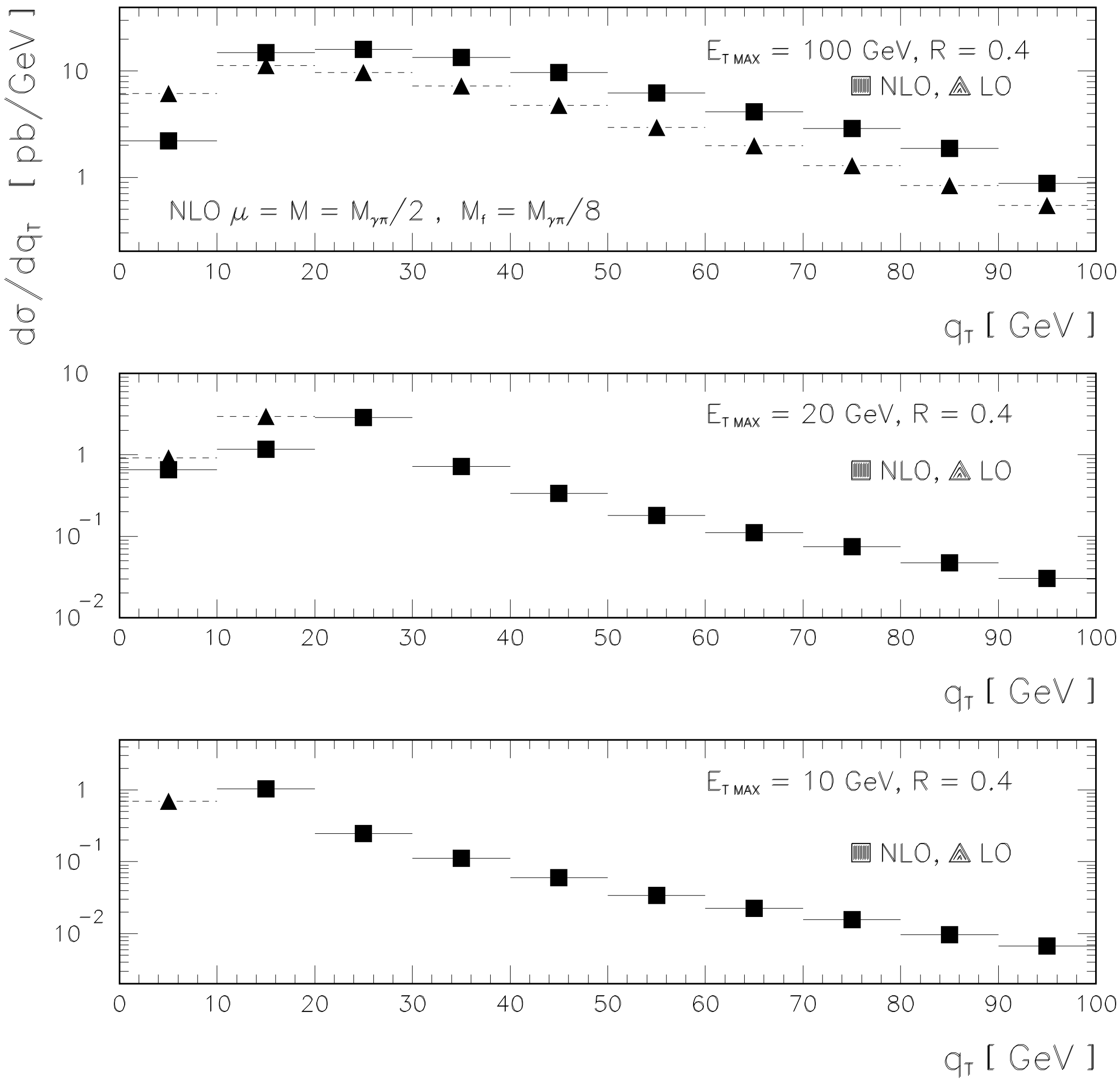
  - numerically robust
  - valid for massless and massive external/internal particles
  - no principal limitation on number of legs
  - optionally more analytic or more numerical branch
  - modular $\Rightarrow$ if more compact expressions for amplitudes can be achieved without using Feynman diagrams: very welcome !
  - real radiation under construction
  - combination with parton shower planned

backup slides

phase space effects enhanced by cuts



N=5 rank 2 example

$$I_5^{n,\,\mu_1\mu_2}(a_1,a_2;S) = g^{\mu_1\,\mu_2}\,B^{5,2}(S) + \sum_{l_1,l_2\in S} \Delta^{\mu_1}_{l_1\,a_1}\,\Delta^{\mu_2}_{l_2\,a_2}\,A^{5,2}_{l_1\,l_2}(S)$$

$$B^{5,2}(S) = -\frac{1}{2}\sum_{j\in S} b_j\, I_4^{n+2}(S\setminus\{j\})$$

$$\begin{aligned}
A^{5,2}_{l_1 l_2}(S) = & \sum_{j\in S}\Big(\mathcal{S}^{-1}{}_{j\,l_1}\,b_{l_2} + \mathcal{S}^{-1}{}_{j\,l_2}\,b_{l_1} \\
& -2\,\mathcal{S}^{-1}{}_{l_1\,l_2}\,b_j + b_j\,\mathcal{S}^{\{j\}-1}{}_{l_1\,l_2}\Big)\; I_4^{n+2}(S\setminus\{j\}) \\
& + \frac{1}{2}\sum_{j\in S}\;\sum_{k\in S\setminus\{j\}}\Big[\mathcal{S}^{-1}{}_{j\,l_2}\,\mathcal{S}^{\{j\}-1}{}_{k\,l_1} + \\
& \mathcal{S}^{-1}{}_{j\,l_1}\,\mathcal{S}^{\{j\}-1}{}_{k\,l_2}\Big]\; I_3^{n}(S\setminus\{j,k\})
\end{aligned}$$

N=6 rank 1 example

$$I_6^{n,\mu}(a;S) = -\sum_{j\in S} \mathcal{C}^{\mu}_{j\,a}\, I_5^n(S\setminus\{j\})$$

$$= -\sum_{j,l\in S} \Delta^{\mu}_{la}\, \mathcal{S}^{-1}_{lj} \sum_{k\in S} b_k^{\{j\}}$$

$$\left[B^{\{j,k\}}\, I_4^{n+2}(S\setminus\{j,k\}) + \sum_{m\in S\setminus\{j,k\}} b_m^{\{j,k\}}\, I_3^n(S\setminus\{j,k,m\}) \right]$$