

Intro to General Relativity

SSF 2005 - Sean Carroll

Metric : $g_{\mu\nu}(t, \vec{x})$

Geodesics : $x^\mu(\lambda)$

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0 \quad (\sum_{\rho,\sigma})$$

$$\Gamma_{\rho\sigma}^\mu = \frac{1}{2} g^{\mu\lambda} (\partial_\rho g_{\sigma\lambda} + \partial_\sigma g_{\lambda\rho} - \partial_\lambda g_{\rho\sigma})$$

Riemann curvature:

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma_{\sigma\nu}^\rho - \partial_\nu \Gamma_{\sigma\mu}^\rho + \Gamma_{\mu\lambda}^\rho \Gamma_{\sigma\nu}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\sigma\mu}^\lambda$$

Ricci Tensor: $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$

Curvature scalar: $g^{\mu\nu} R_{\mu\nu} = R$

Einstein's Eq: $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$

$T_{\mu\nu}$ = energy-momentum tensor

Outline

- Overview
- GR is a field theory
- Equivalence principle
- The metric
- Geodesics
- Spacetime
- Non-flat metrics
- Vectors
- Tensors
- Non-tensors
- Covariant derivatives
- Gauge theory
- Curvature
- Einstein's equation
- Matter
- Newtonian limit
- Cosmology
- Homework

GR is a field theory

2 ways to get GR:

- "Gravity" = curvature of space time

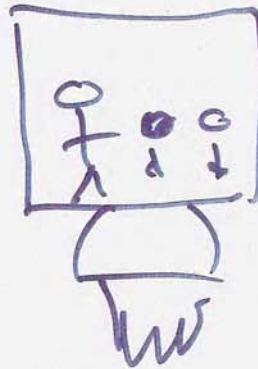
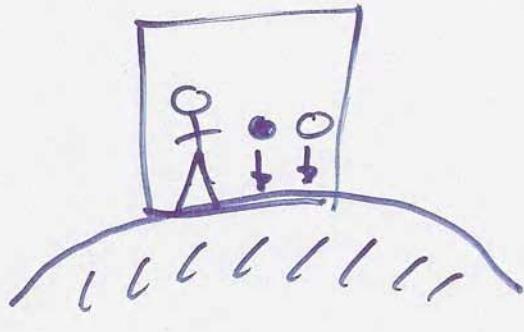
metric tensor: $g_{\mu\nu}(t, \vec{x}) = g_{\mu\nu}$

- massless spin-2 particles

$$h_{\mu\nu} = h_{\nu\mu}$$

$$\rightarrow g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(h^2)$$

Principle of Equivalence

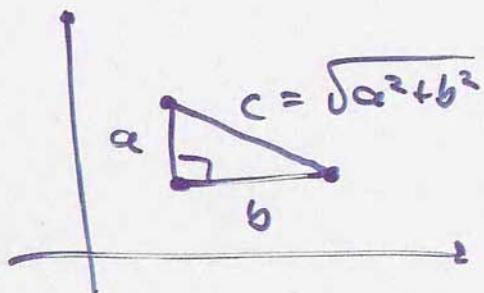


→ Gravity is universal

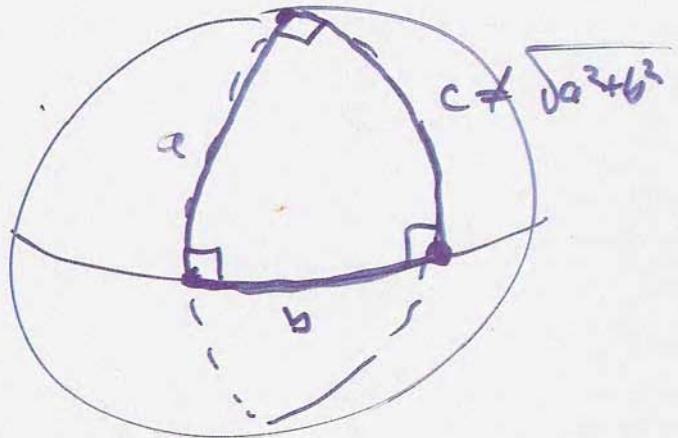
→ ∴ it is not a "force"

Metric

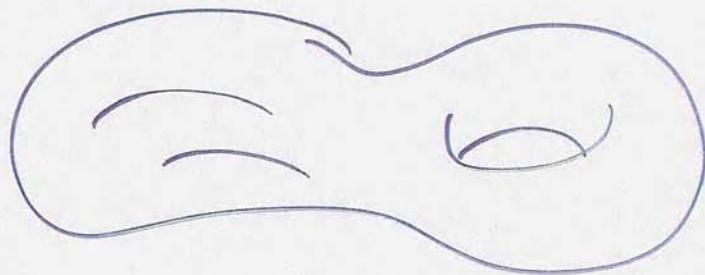
Flat:



Curved:



Curvature need not
be uniform:



Go infinitesimal:

$$\frac{dx}{dy} \Delta s^2 = dx^2 + dy^2$$

Curvature:

$$ds^2 = f(x,y) dx^2 + g(x,y) dx dy + h(x,y) dy^2$$

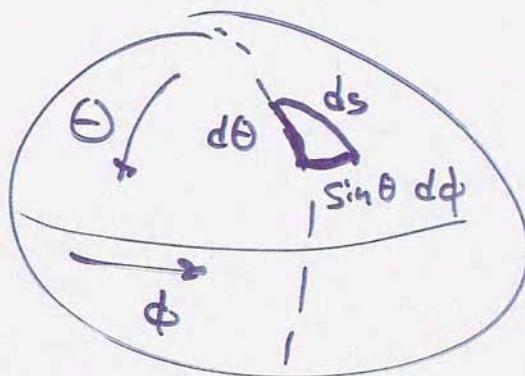
Better notation:

$$ds^2 = \sum_{i,j} g_{ij}(x) dx^i dx^j$$

$$x^i = \{x, y\}$$

$$ds^2 = g_{ij} dx^i dx^j \rightarrow \text{line element}$$

Sphere:

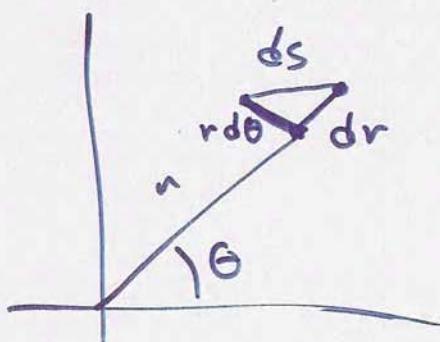


$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2$$

$$= g_{ij} dx^i dx^j$$

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

Plane in polar coordinates:

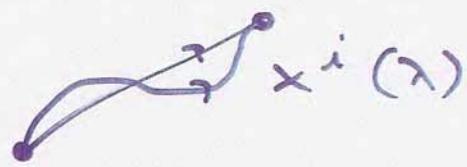


$$ds^2 = dr^2 + r^2 d\theta^2$$

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Geodesics : minimum-length paths

Integral:



$$I = \int ds = \int \sqrt{ds^2} = \int \sqrt{g_{ij} dx^i dx^j}$$

"Velocity": $v^i = \frac{dx^i}{d\lambda}$

$$\left(\frac{ds}{d\lambda} \right)^2 = g_{ij} \left(\frac{dx^i}{d\lambda} \right) \left(\frac{dx^j}{d\lambda} \right) = g_{ij} v^i v^j = \vec{v} \cdot \vec{v}$$

$$I = \int \sqrt{g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}} d\lambda$$

Vary the path: $x^i(\lambda) \rightarrow x^i(\lambda) + \delta x^i(\lambda)$

$$I \rightarrow I + \delta I$$

$$\frac{dx^i}{d\lambda} \rightarrow \frac{dx^i}{d\lambda} + \frac{d(\delta x^i)}{d\lambda}$$

$$g_{ij}(x^k) \rightarrow g_{ij}(x^k + \delta x^k)$$

$$g_{ij} = g_{ij}(x^k) + \frac{\partial g_{ij}}{\partial x^k} \delta x^k$$

$$= g_{ij}(x^k) + \partial_k g_{ij} \delta x^k$$

$$I = \int ds = \int \sqrt{g_{ii} \frac{dx^i}{d\lambda} \frac{dx^i}{d\lambda}} d\lambda$$

$$x^i - x^i + \delta x^i$$

$$H = I + \delta I$$

$$\delta I = 0$$

$$\rightarrow \frac{d^2 x^k}{d\lambda^2} + \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij}) \frac{dx^i}{d\lambda} \frac{dx^l}{d\lambda} = 0$$

g^{kl} = inverse metric

$$g^{kl} g_{li} = \delta_i^k = 1$$

$$\text{Let } \Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij})$$

→ Christoffel Symbols
or Connection Coefficients

$$\text{Flat : } ds^2 = dx^2 + dy^2 \Rightarrow g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\frac{d^2 x^k}{d\lambda^2} + \Gamma_{ij}^k \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} = 0} \quad \lambda \rightarrow a\lambda + b$$

timelike paths: $ds^2 < 0$

Proper time: $d\tau^2 = -ds^2$

> 0 timelike paths

$$\tau = \int \sqrt{d\tau^2} = \int \sqrt{-ds^2} = \int \sqrt{-g_{\mu\nu} dx^\mu dx^\nu}$$

$$= \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

τ is the time measured by
clocks moving along $x^\mu(\lambda)$.

Geodesics: $\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0$

→ Maximize the proper time

Newtonian limit (static, weak)

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)[dx^2 + dy^2 + dz^2]$$

Φ = gravitational potl. $(\nabla^2 \Phi = 4\pi G\rho)$

Geodesics: $\frac{d^2x^\mu}{d\tau^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0$

particles moving slowly: $\left| \frac{dx^i}{d\tau} \right| \ll \left| \frac{dx^0}{d\tau} \right|$

$$\tau \approx t$$

$\mu=i$: $\frac{d^2x^i}{dt^2} + \Gamma_{\rho\sigma}^i \frac{dx^\rho}{dt} \frac{dx^\sigma}{dt} = 0$

$$\rightarrow \frac{d^2x^i}{dt^2} + \Gamma_{00}^i \left(\frac{dx^0}{dt} \right)^2 = 0$$

$$\Gamma_{00}^i = \frac{1}{2} g^{i\lambda} (\partial_0 g_{0\lambda} + \partial_\lambda g_{00} - \partial_0 g_{00})$$

$$= \frac{1}{2} g^{ii} (-\partial_i g_{00})$$

$$= \frac{1}{2} (1) (-\partial_i (-2\Phi)) = \partial_i \Phi$$

$$\rightarrow \boxed{\frac{d^2x^i}{dt^2} = -\partial_i \Phi}, \vec{F} = m\vec{a} = -m\vec{\nabla}\Phi$$

Vectors

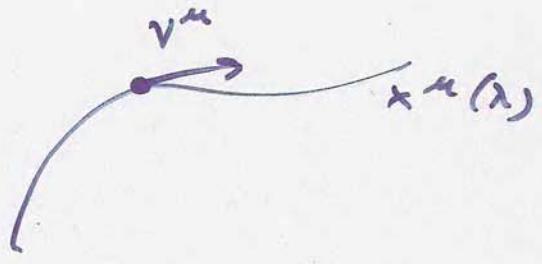
4-vectors

$$V^\mu = (V^0, V^1, V^2, V^3)$$



- Tangent vector to a curve $x^\mu(\lambda)$:

$$V^\mu = \frac{dx^\mu}{d\lambda}$$



$$g_{\mu\nu} V^\mu V^\nu < 0 \Rightarrow \text{timelike}$$

- 4-velocity $U^\mu = \frac{dx^\mu}{d\tau} = \frac{d\lambda}{d\tau} \frac{dx^\mu}{d\lambda}$

$$\begin{aligned} g_{\mu\nu} U^\mu U^\nu &= g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \frac{(g_{\mu\nu} dx^\mu dx^\nu)}{d\tau^2} \\ &= \frac{(-d\tau^2)}{d\tau^2} = -1 \end{aligned}$$

- 4-momentum $P^\mu = m U^\mu$

straight-line in Minkowski: $\gamma = \frac{1}{\sqrt{1-v^2}}$

$$\begin{matrix} P^\mu = (m\gamma, m\gamma \vec{v}) \\ \downarrow \quad \downarrow \\ E \quad \vec{P} \end{matrix}$$

$$\vec{v}=0 \Rightarrow \gamma=1 \Rightarrow E=m$$

Tensors :

$g_{\mu\nu}$, $g^{\mu\nu}$, p^μ are all tensors :

collections of functions that
transform correctly

e.g. $U^\mu = \frac{dx^\mu}{dt}$ $x^{\mu'} = x^{\mu'}(x^\mu)$

$$dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} dx^\mu$$

note: $\frac{\partial x^\mu}{\partial x^{\mu'}}$ is the inverse of $\frac{\partial x^{\mu'}}{\partial x^\mu}$:

$$\frac{\partial x^{\mu'}}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^{\mu'}} = \delta_\nu^\mu$$

$$\boxed{U^{\mu'}} = \frac{dx^{\mu'}}{dt} = \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{dx^\mu}{dt} = \boxed{\frac{\partial x^{\mu'}}{\partial x^\mu} U^\mu}$$

Vectors: $U^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} U^\mu$

Metric: ~~$g_{\mu\nu}$~~ $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu'\nu'} dx'^\mu dx'^\nu$

divide by $dx'^\mu dx'^\nu$:

$$g_{\mu'\nu'} = g_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\mu} \frac{\partial x^\nu}{\partial x'^\nu}$$

Tensor transformation law:

$$S^{\alpha\beta\gamma}_{\omega'} = \frac{\partial x^\alpha}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x^\beta} \frac{\partial x^\gamma}{\partial x^\gamma} \frac{\partial x^\omega}{\partial x^\omega} S^{\alpha\beta\gamma}_\omega$$

Index slot is important:

$$S^{\alpha\beta\gamma}_\sigma = g_{\sigma\beta} S^{\alpha\beta\gamma}_\omega$$

Contract indices:

$$S^{\alpha\beta} = S^{\alpha\beta\lambda}_\lambda$$

$$\tilde{S}^{\alpha\beta} = S^{\alpha\beta\lambda}_\lambda$$

~~$S^{\alpha\beta\gamma}_\omega$~~

Inner product: $g_{\mu\nu} A^\mu B^\nu$

"Lowering indices": $A_\nu = g_{\mu\nu} A^\mu$

$$g_{\mu\nu} A^\mu B^\nu = A_\nu B^\nu$$

"Raising": $g^{\rho\sigma} A_\sigma = A^\rho$

$$g_{\mu\nu} V^\nu = g_{\mu\nu} (g^{\nu\sigma} V_\sigma)$$

$$[g_{\mu\nu} g^{\nu\sigma} = \delta_\mu^\sigma]$$

$$g_{\mu\nu} V^\nu = \delta_\mu^\sigma V_\sigma = V_\mu \quad \checkmark$$

Symmetric: $S^{\alpha\beta\gamma}_\omega = S^{\beta\alpha\gamma}_\omega$

Antisymmetric: $S^{\alpha\beta\gamma}_\omega = -S^{\beta\alpha\gamma}_\omega$

e.g. $g_{\mu\nu} = g_{\nu\mu}$ is symmetric

$F_{\mu\nu} = -F_{\nu\mu}$ is anti-

$$= \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

Add: $A_{\alpha\beta} + B_{\alpha\beta} = C_{\alpha\beta}$ $(k,l) + (k,l) = (k,l)$

(k,l) : $T^{M_1 M_2 \dots M_k}_{\quad \quad \quad v_1 \dots v_l}$

Multiply: $A^\alpha B^\beta \gamma = C^{\alpha\beta} \gamma$

$(k,l) \otimes (m,n) = (k+m, l+n)$

$\Delta ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad T_{\alpha\beta} = T_{(\alpha\beta)} + T_{[\alpha\beta]}$

Minkowski Spacetime

- $x^i \rightarrow x^\mu = \{x^0, x^1, x^2, x^3\}$

$$= \{t, x, y, z\}$$

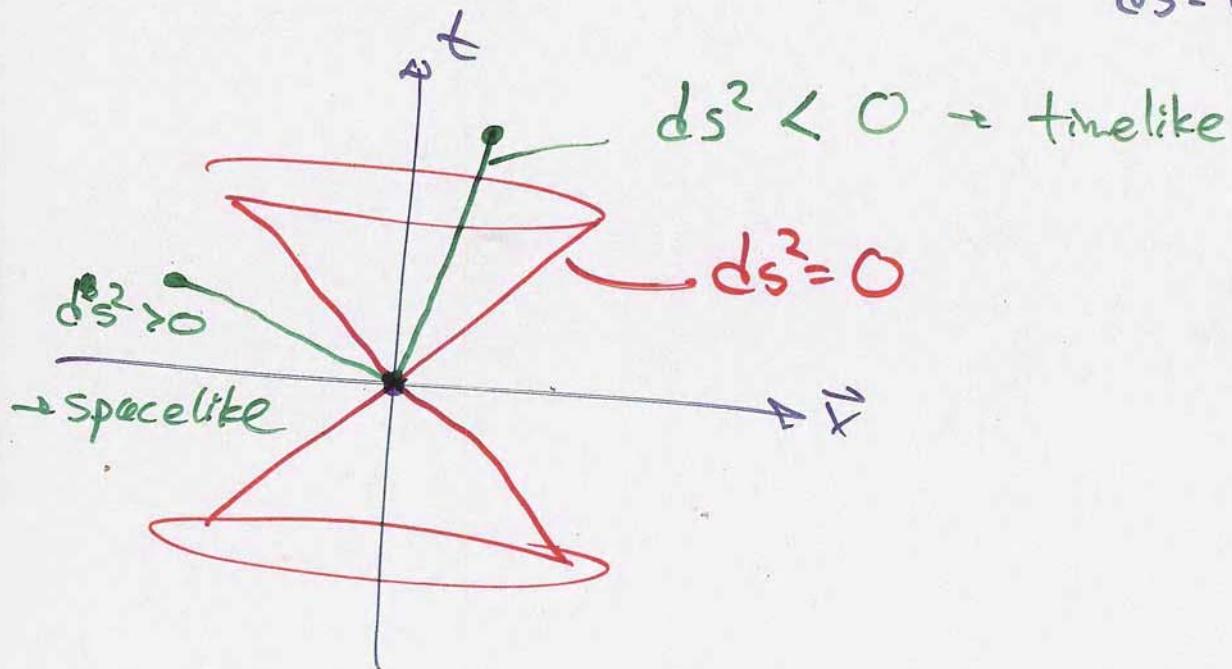
- minus signs

Metric: $\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}$

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$= \eta_{\mu\nu} dx^\mu dx^\nu$$

$ds^2 = 0 \rightarrow$ lightlike
(null)



Non - flat metrics

- Expanding universe :

$$ds^2 = -dt^2 + a^2(t) \underbrace{[dx^2 + dy^2 + dz^2]}_{\text{or } S^3} \quad \text{or } H^3$$
$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & a^2 & & \\ & & a^2 & \\ & & & a^2 \end{pmatrix}$$

- Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$
$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

Birkhoff's Theorem:

Schwarzschild is the unique spherically symmetric solution to Einstein's Eq. in vacuum.

Non-Tensors

① Partial derivatives

Gradient: $\partial_\mu \phi = \frac{\partial \phi}{\partial x^\mu}$

$$\partial_{\mu'} \phi = \frac{\partial \phi}{\partial x^{\mu'}} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial \phi}{\partial x^\mu} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \phi$$

(0,1) tensor = dual vector

But $\partial_\mu V^\nu = \frac{\partial V^\nu}{\partial x^\mu}$

$$\partial_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \left(\frac{\partial x^{\nu'}}{\partial x^\nu} V^\nu \right)$$

$$= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} V^\nu \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu}$$

q.e.d.

② Christoffel symbols

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\mu\lambda} (\partial_\rho g_{\sigma\lambda} + \partial_\sigma g_{\lambda\rho} - \partial_\lambda g_{\rho\sigma})$$

$$\Gamma^\mu_{\rho\sigma} = \underbrace{\quad\quad\quad}_{\quad\quad\quad}$$

$$= \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^\rho}{\partial x^{\rho'}} \frac{\partial x^\sigma}{\partial x^{\sigma'}} \Gamma^\mu_{\rho\sigma} - \frac{\partial x^\rho}{\partial x^{\rho'}} \frac{\partial x^\sigma}{\partial x^{\sigma'}} \frac{\partial^2 x^{\mu'}}{\partial x^\mu \partial x^{\rho'\sigma'}}$$

③ Determinant of $g_{\mu\nu}$

$$g = |g_{\mu\nu}| = \text{Det } g_{\mu\nu}$$

$$x^\mu \rightarrow x^{\mu'} : \quad g' = \left| \frac{\partial x^\mu}{\partial x^{\mu'}} \right|^2 g$$

+ tensor density

④ Volume element $d^n x = dx^0 dx^1 \dots dx^{n-1}$

$$d^n x' = \left| \frac{\partial x^{\mu'}}{\partial x^\mu} \right| d^n x$$

Tensorial volume element

$$\sqrt{-g} d^n x = \sqrt{-g'} d^n x'$$

$$X^{\alpha B}{}_\gamma = Y^{\alpha B}{}_\gamma$$

$$\rightarrow X^{\alpha' B'}{}_{\gamma'} = Y^{\alpha' B'}{}_{\gamma'}$$

$$\partial_\mu j^\mu = 0$$

Covariant Derivatives

Recall: $\partial_\mu V^\nu = \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\lambda} \partial_\lambda V^\nu + \frac{\partial x^\mu}{\partial x^\nu} V^\nu \frac{\partial^2 x^\nu}{\partial x^\mu \partial x^\lambda}$

$$\boxed{\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\lambda}^\nu V^\lambda}$$

$$\nabla_\mu V^\nu = \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\lambda} \nabla_\lambda V^\nu$$

Dual vectors:

$$\nabla_\mu w_\nu = \partial_\mu w_\nu - \Gamma_{\mu\nu}^\lambda w_\lambda$$

$$\nabla_\mu S^\alpha{}_\beta = \partial_\mu S^\alpha{}_\beta + \Gamma_{\mu\lambda}^\alpha S^\lambda{}_\beta - \Gamma_{\mu\beta}^\lambda S^\alpha{}_\lambda$$

Metric Compatibility:

$$\nabla_\mu g_{\rho\sigma} = 0 = \nabla_\mu g^{\rho\sigma}$$

$$\nabla_\mu (AB) = (\nabla_\mu A)B + A(\nabla_\mu B)$$

Gauge Theory

GR

v^μ

$$v^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} v^\mu$$

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \underline{\Gamma_{\mu\nu}^\nu} v^\lambda$$

E+M

ψ

$$\psi' = e^{i\alpha} \psi$$

$$D_\mu \psi = \partial_\mu \psi - ie A_\mu \psi$$

$$A_\mu \rightarrow A_\mu + \partial_\mu \lambda$$

$$D_\mu \psi \rightarrow e^{i\lambda} D_\mu \psi$$

Field strength : $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$

$$\begin{aligned} [D_\mu, D_\nu] \psi &= D_\mu D_\nu \psi - D_\nu D_\mu \psi \\ &= F_{\mu\nu} \psi \end{aligned}$$

Curvature Tensor

$$[\nabla_\mu, \nabla_\nu] V^\rho = \nabla_\mu (\nabla_\nu V^\rho) - \nabla_\nu (\nabla_\mu V^\rho) \\ = R^\rho_{\sigma\mu\nu} V^\sigma$$

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\sigma\nu} - \partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\sigma\nu} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\sigma\mu}$$

↳ Riemann Curvature Tensor

Properties: $R^\rho_{\sigma\mu\nu} = -R^\rho_{\sigma\nu\mu}$

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^\lambda_{\sigma\mu\nu}$$

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$$

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$$

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0$$

~~$$R^\rho_{\sigma\mu\nu} = -R^\rho_{\mu\nu\sigma}$$~~

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\lambda\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} = 0$$

"Bianchi Identity"

$$\partial_\mu F_{\rho\sigma} + \partial_\rho F_{\sigma\mu} + \partial_\sigma F_{\mu\rho} = 0$$

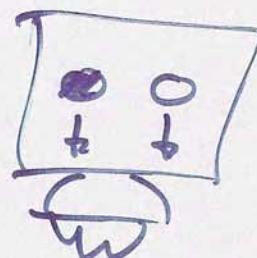
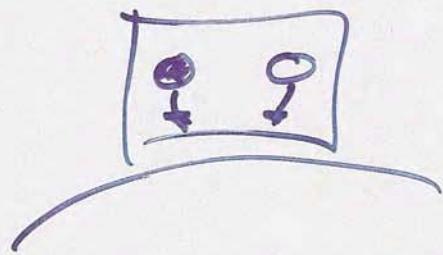
Contractions of $R^\rho_{\sigma\mu\nu}$:

Ricci tensor: $R_{\sigma\nu} = R^\lambda_{\sigma\lambda\nu}$

Curvature scalar: $R = g^{\sigma\nu} R_{\sigma\nu}$

Bianchi Identity $\rightarrow \nabla^\mu (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) = 0$

Physics? $R^\rho_{\sigma\mu\nu} \rightarrow$ tidal forces



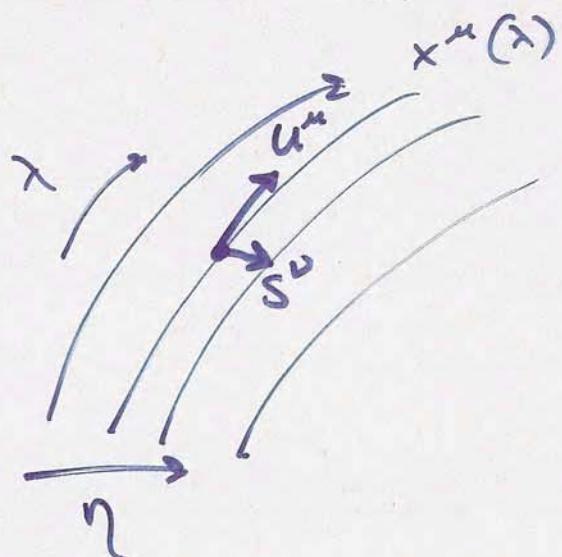
Geodesic Deviation

$$U^\mu = \frac{dx^\mu}{d\lambda} \text{ at } \eta = \text{const.}$$

$$S^\nu = \frac{dx^\nu}{d\eta} \text{ at } \lambda = \text{const}$$

$$\boxed{\frac{D^2}{d\lambda^2} S^\nu = R^\rho_{\sigma\mu\nu} U^\sigma U^\mu S^\nu}$$

$$\frac{D}{d\lambda} = \frac{dx^\lambda}{d\lambda} \nabla_\alpha$$



Einstein's Equation

Vary an action S_E wrt $g_{\mu\nu}(x)$

$$S = \int d^4x \sqrt{-g} \mathcal{L} = \int d^4x \sqrt{-g} R$$

$$g^{\mu\nu} g_{\mu\nu} = 4$$

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$$

$$R = \underline{g^{\mu\nu}} R_{\mu\nu}$$

$$\delta S = \int d^4x \sqrt{-g} (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) \delta g^{\mu\nu}$$

$$\Rightarrow R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$

in vacuum

$$\nabla^2 \underline{g} = 0$$

$$\text{Contract: } R - \frac{1}{2} R (4) = -R = 0$$

$$\rightarrow \boxed{R_{\mu\nu} = 0}$$

$$\text{Check: } g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2GM}{r}\right) & \\ & \left(1 - \frac{2GM}{r}\right)^{-1} \end{pmatrix}_{r^2} \quad r^2 \sin^2\theta$$

Matter

$$S = \int d^4x \sqrt{g} \left[\frac{R}{16\pi G} + \mathcal{L}_M \right]$$

$$S \rightarrow S + SS \quad \text{as } g^{\mu\nu} \rightarrow g^{\mu\nu} + \delta g^{\mu\nu}$$

$$\frac{1}{16\pi G} \left[R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right] + \frac{1}{\sqrt{g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} = 0$$

$$T_{\mu\nu} = -2 \frac{1}{\sqrt{g}} \frac{\delta S_m}{\delta g^{\mu\nu}} \rightarrow \begin{array}{l} \text{Energy} \\ \text{Momentum} \\ \text{Tensor} \end{array}$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\nabla^\mu (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) = 0 \rightarrow \nabla^\mu T_{\mu\nu} = 0$$

$$\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\mathcal{L}_\phi = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi)$$

Cosmology

Flat RW: $ds^2 = -dt^2 + a^2(t) [dx^2 + dy^2 + dz^2]$

$$g_{\mu\nu} = \begin{pmatrix} -1 & a^2 & a^2 & a^2 \\ a^2 & a^2 & a^2 & a^2 \end{pmatrix} \quad g^{\mu\nu} = \begin{pmatrix} -1 & a^{-2} & a^{-2} & a^{-2} \\ a^{-2} & a^{-2} & a^{-2} & a^{-2} \end{pmatrix}$$

$$g_{00} = -1, \quad g_{ij} = a^2 \delta_{ij} \quad g^{00} = -1, \quad g^{ij} = a^{-2} \delta^{ij}$$

$$\Gamma_{\mu\nu}^\rho = \frac{1}{2} g^{\rho\lambda} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})$$

$$\Gamma_{ij}^0 = \frac{1}{2} g^{0\lambda} (\cancel{\partial}_i g_{j\lambda} + \cancel{\partial}_j g_{i\lambda} - \cancel{\partial}_\lambda g_{ij})$$

$$= \frac{1}{2} g^{00} (-\partial_0 g_{ij})$$

$$= \frac{1}{2} (-1) (-\partial_0 a^2 \delta_{ij})$$

$$= \frac{1}{2} (-1) (-2) \dot{a} a \delta_{ij}$$

$$\rightarrow \Gamma_{ij}^0 = \dot{a} a \delta_{ij}$$

$$\Gamma_{j0}^i = \frac{\dot{a}}{a} \delta_j^i$$

$$\text{all others} = 0$$

Perfect Fluids

$$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu}$$

ρ = energy density

p = pressure

U^μ = 4-velocity of fluid

Choose coords. @ point P s.t.

$$g_{\mu\nu}(P) = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}$$

And $U^\mu = (1, 0, 0, 0)$

$$T_{\mu\nu} = \begin{pmatrix} \rho & \\ & p \delta_{ij} \end{pmatrix}$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & \alpha^2 & \alpha^2 & \alpha^2 \\ \alpha^2 & \alpha^2 & \alpha^2 & \alpha^2 \\ \alpha^2 & \alpha^2 & \alpha^2 & \alpha^2 \\ \alpha^2 & \alpha^2 & \alpha^2 & \alpha^2 \end{pmatrix} \quad g^{\mu\nu} = \begin{pmatrix} -1 & \alpha^{-2} & \alpha^{-2} & \alpha^{-2} \\ \alpha^{-2} & \alpha^{-2} & \alpha^{-2} & \alpha^{-2} \\ \alpha^{-2} & \alpha^{-2} & \alpha^{-2} & \alpha^{-2} \\ \alpha^{-2} & \alpha^{-2} & \alpha^{-2} & \alpha^{-2} \end{pmatrix}$$

$$\underline{T}_{\mu\nu}^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu}$$

$$u^\mu = (1, 0, 0, 0)$$

$$\text{check: } g_{\mu\nu} u^\mu u^\nu = -1$$

$$\underline{T}^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & \alpha^{-2} p & \delta^{ij} & \end{pmatrix}$$

$$\nabla_\mu T^{\mu\nu} = 0 \quad \delta^{i\bar{i}} \delta_{j\bar{j}} = 3$$

$$\underline{v=0}: \quad \nabla_\mu T^{\mu 0} = 0$$

$$= \partial_\mu T^{\mu 0} + \Gamma_{\mu\lambda}^\lambda T^{\lambda 0} + \Gamma_{\mu\lambda}^0 T^{\mu\lambda}$$

$$= \partial_0 T^{00} + \sum_i \Gamma_{i0}^i T^{00} + \sum_i \Gamma_{ii}^0 T^{ii}$$

$$= \partial_0(p) + 3 \frac{\dot{\alpha}}{\alpha} p + 3 \frac{\dot{\alpha}}{\alpha} p = 0$$

$$\left. \dot{p} = -3 \frac{\dot{\alpha}}{\alpha} (p + p) \right\}$$

$$\begin{aligned} p &= w p \\ \Rightarrow p &\propto \alpha^{-3(1+w)} \end{aligned}$$

$$ds^2 = -dt^2 + \alpha^2(t) (dx^2)$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$R_{\sigma\nu} = R^\lambda_{\sigma\lambda\nu} = \partial_\nu \Gamma^\lambda_{\sigma\nu} - \partial_\sigma \Gamma^\lambda_{\sigma\lambda} + \Gamma^\lambda_{\lambda\omega} \Gamma^{\omega}_{\sigma\nu} - \Gamma^\lambda_{\omega\sigma} \Gamma^{\omega}_{\nu\lambda}$$

$$R_{00} = \cancel{\partial_0 \Gamma^\lambda_{00}} - \cancel{\partial_0 \Gamma^\lambda_{0\lambda}} + \cancel{\Gamma^\lambda_{\lambda\omega} \Gamma^{\omega}_{00}} - \cancel{\Gamma^\lambda_{0\omega} \Gamma^{\omega}_{0\lambda}}$$

$$= -\partial_0 \left(3 \frac{\dot{a}}{a} \right) + 0 - \left(\frac{\dot{a}}{a} \right)^2 S_1^1 S_1^1$$

$$= -3 \partial_0 \left(\frac{\dot{a}}{a} \right) - 3 \left(\frac{\dot{a}}{a} \right)^2$$

$$= -3 \left[\ddot{\frac{a}{a}} - \left(\frac{\dot{a}}{a} \right)^2 \right] - 3 \left(\frac{\dot{a}}{a} \right)^2$$

$$= -3 \ddot{\frac{a}{a}}$$

$$\frac{\dot{a}}{a} = H$$

$$R_{ij} = (\ddot{a}a + 2\dot{a}^2) \delta_{ij} \quad P \propto a^{-n}$$

$$R = 6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \rightarrow a \propto t^{2/n}$$

$$R_{00} - \frac{1}{2} R g_{00} = 3 \frac{\dot{a}^2}{a^2} = 8\pi G T_{00}$$

$$\Rightarrow \boxed{\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} P} \quad \text{Friedmann Equation}$$